Energy Conserving and Projection Methods for the Real-Time Dynamics of Multibody Systems

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Abstract— *Among the different methods proposed for real-time dynamics of multibody systems, penalty and augmented Lagrangian techniques have been widely used, in order to convert the system of differential algebraic equations (DAE) into a system of ordinary differential equations (ODE), whose integration presents a lower level of difficulty. In both cases, penalty forces which are proportional to the violation of the constraints at position, and/or velocity, and/or acceleration level, are included in the equations of motion in order to ensure the satisfaction of the constraints and, if possible, of their derivatives.*

In principle, consideration of penalty forces which are only proportional to the violation of the constraints at position level is the most attractive option, since the velocity level implies energy dissipation, and the acceleration level is computationally expensive. However, the solutions obtained through such option show unstable behavior due to the progressive and unbounded growth of the constraints energy, effect even more acute in the case of augmented Lagrangian formulations.

Several methods have been proposed in the literature in order to stabilize the mentioned solutions. In a previous work, two of those methods were revised and their respective stabilizing properties compared: an energy-momentum integrator, and the trapezoidal rule along with mass-orthogonal projections of velocities and accelerations onto the constraints manifold. In both cases, natural coordinates were used for the modeling, and the equations of motions were stated through the augmented Lagrangian formulation. In this work, the same two methods are compared in terms of efficiency by simulating a large and realistic problem.

Keywords: multibody systems, real-time integration

I. Introduction

Several considerations are important if we try to carry out fast and precise simulations in multibody dynamics: the choice of modeling coordinates, the choice of the dynamical formulation, and the numerical integration scheme along with the numerical implementation. All these matters are essential in order to decide whether a specific method is suitable for a particular purpose.

Some of the most robust methods for the real-time dynamics of multibody systems make use of natural or fully Cartesian coordinates [9], which are dependent by nature, and lead to systems of differential-algebraic equations of motion (DAE) [5].

In order to solve such DAE in natural coordinates, different formulations have been developed, like Baumgarte stabilization [1], penalty and augmented Lagrangian schemes [2], or velocity transformations [18], [16].

Formulations based on penalty and augmented Lagrangian methods have the advantages of being very simple, computationally inexpensive and very robust in the presence of singular configurations or redundant constraints [4].

Generally, it can be said that the choice of the dynamic formulation determines that of the numerical integrator. In this direction, different authors proposed several options to successfully integrate the equations arising from constrained multibody systems, using integrators coming from the field of structural dynamics [9], [11], [6].

In [4], [6], the use of augmented Lagrangian techniques with penalty only at position level along with the trapezoidal rule was proposed. In order to guarantee the correct satisfaction of the constraints, velocity and acceleration projections were proposed. More recently, [8] proposed the use of augmented Lagrangian techniques with other integrators of the Generalized-α family along with projections, obtaining very good behavior for real-time applications. The advantages of the projections are the simplicity and that they can be used with a great variety of integrators. The projections are responsible for maintaining the stability of the formulation.

On the other hand, other authors, [12], [10], [13], developed a formulation based on an energy conserving penalty scheme, enforcing the constraints at the position level, and

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applied it to the dynamics of multibody systems parametrized with natural coordinates. In this case, the use of penalty at position level has the advantage of enabling to derive the constraint forces from a potential function: the constraint energy. The formulation employs an energy-momentum integrator as integration scheme [17], [14], so that the conservation of the total energy of the system is imposed by construction of the algorithm. Here, the stabilization of the penalty equations of motion arises in a natural manner from the integration scheme.

II. Augmented Lagrangian Formulation

Many different methods have been proposed in the literature for the dynamics of constrained mechanical systems. The formulation of the equations of a constrained mechanical system poses some numerical difficulties. These difficulties are, in general, different for each formulation and solution method, but are typically related to stability properties of the numerical scheme. Such problems motivate the interest in developing algorithms capable of providing stable and accurate solutions.

In this work, we restrict ourselves to the methods based on the augmented Lagrangian formulation, which can be understood as a compromise between the Lagrange multiplier formulation, and the penalty formulations.

A. Description of the formulation

Let us consider a multibody system, with a configuration defined by a vector $\mathbf{q} \in \Re^n$. The system is also subjected to *m* holonomic constraints $\mathbf{\Phi} \in \Re^m$ involving the different points and vectors of the system. The dynamic equations constitute an index-3 DAE system of $n+m$ equations given by:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_\mathbf{q}^\mathrm{T}\alpha\mathbf{\Phi} + \mathbf{\Phi}_\mathbf{q}^\mathrm{T}\boldsymbol{\lambda}^* = \mathbf{Q} \quad , \quad \mathbf{\Phi} = \mathbf{0} \qquad (1)$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{Q}$ is the generalized forces vector of the system, $\boldsymbol{\lambda}$ represents the Lagrange multiplier vector, α is a penalty factor, and $\mathbf{\Phi_q}$ is the Jacobian matrix of the constraints vector.

In practice, the augmented Lagrangian formulation, transforms the DAE into a system of ordinary differential equations (ODE), defining an iterative update for the multipliers, given by $\boldsymbol{\lambda}^*_{i+1} = \boldsymbol{\lambda}^*_i + \alpha\mathbf{\Phi}$ verifiying that $\boldsymbol{\lambda}^* \rightarrow \boldsymbol{\lambda}$ as $i \rightarrow \infty$, which means that, in the limit, the iterative scheme for the Lagrange multipliers leads to the true Lagrange multipliers. Moreover, the iteration update for the multipliers, prevents introducing the Lagrange multipliers as unknowns of the problem, so that the system of $n + m$ equations is replaced by another one of size only n.

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_\mathbf{q}^\mathrm{T}\alpha\mathbf{\Phi} + \mathbf{\Phi}_\mathbf{q}^\mathrm{T}\boldsymbol{\lambda}^* = \mathbf{Q} \quad , \quad \boldsymbol{\lambda}^*_{i+1} = \boldsymbol{\lambda}^*_i + \alpha\mathbf{\Phi} \qquad (2)$$

In order to solve the nonlinear equations (2), the use of Newton-Raphson schemes is recommended, and then it is possible and convenient to mix up the Newton-Raphson iteration and the Lagrange multipliers iteration. Several authors have used methods based on the equations (2) for the dynamics of multibody systems. This equations show a slightly unstable behavior with most of the commonly used integrators, characterized by the drift-off effect and a progressive and unbounded growth of the total energy of the system.

III. Index-3 augmented Lagrangian with projections

This formulation was presented in [6], [7]. It is based on the an augmented Lagrangian equations (2).

The augmented Lagrangian formulation can be combined with any standard integrator, and achieves the exact fulfillment of the position constraints, but usually exhibits an unstable behavior, even with ODE integrators suited to stiff systems.

Based on previous results obtained in the literature, it can be said that the numerical solution of a constrained mechanical system seems to be more stable on the constraints manifold than off it. Based on this fact, the exact enforcement of the constraints at velocity and acceleration levels ($\dot{\mathbf{\Phi}} = \mathbf{0}$ and $\ddot{\mathbf{\Phi}} = \mathbf{0}$ respectively), which is not accomplished by the augmented Lagrangian formulation (2), is expected to stabilize the numerical solution.

The way to enforce the constraints proposed in this section is the so-called coordinate projection technique. The velocities and accelerations coming from the integrator, are projected onto the constraints manifolds $\dot{\mathbf{\Phi}} = \mathbf{0}$ and $\ddot{\mathbf{\Phi}} = \mathbf{0}$ respectively, by means of the following expressions:

$$(\mathbf{P} + \mathbf{\Phi}_\mathbf{q}^\mathrm{T} k\alpha\mathbf{\Phi_q})\dot{\mathbf{q}} = \mathbf{P}\dot{\mathbf{q}}^* - \mathbf{\Phi}_\mathbf{q}^\mathrm{T} k\alpha\mathbf{\Phi}_t \qquad (3)$$

$$(\mathbf{P} + \mathbf{\Phi}_\mathbf{q}^\mathrm{T} k\alpha\mathbf{\Phi_q})\ddot{\mathbf{q}} = \mathbf{P}\ddot{\mathbf{q}}^* - \mathbf{\Phi}_\mathbf{q}^\mathrm{T} k\alpha(\dot{\mathbf{\Phi}}_\mathbf{q}\dot{\mathbf{q}} + \dot{\mathbf{\Phi}}_t) \qquad (4)$$

where $\mathbf{P}$ is the projection matrix, $\dot{\mathbf{q}}^*$ and $\ddot{\mathbf{q}}^*$ are the velocities and accelerations coming from the integrator, $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ are the projected velocities and accelerations, and k is a real constant.

The choice of the projection matrix $\mathbf{P}$ is of key importance, since it determines the numerical stability of the solution. Moreover, a correct choice of this matrix can guarantee the unconditional energy-dissipative character of the projections, which proves to be very efficient for stabilizing the system.

By a careful choice of the projection matrix $\mathbf{P}$ and the real constant k, it is also possible to equal the LHS of the linear systems (3) and (4), $(\mathbf{P} + \mathbf{\Phi}_\mathbf{q}^\mathrm{T} k\alpha\mathbf{\Phi_q})$, to the tangent matrix of the Newton-Raphson iteration for the nonlinear system (2), matrix which must be factorized only once, thus saving much time at the projection stage.

IV. Conserving augmented Lagrangian formulation

The point of departure of the approach described in this section, is the energy-momentum formulation presented in

[10], [12], [13]. The menctioned energy-momentum formulation, was based on the following penalty equations:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}}\alpha\mathbf{\Phi} = \mathbf{Q} \tag{5}$$

On the other hand, the formulation described in this section, is based on the equations (2), repeated here for clarity:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}}\alpha\mathbf{\Phi} + \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\lambda}^{*} = \mathbf{Q} \quad , \quad \boldsymbol{\lambda}_{i+1}^{*} = \boldsymbol{\lambda}_{i}^{*} + \alpha\mathbf{\Phi} \tag{6}$$

The key point of the conserving approach is the formulation of the constraint and conservative forces in such a way that guarantees the algorithmic conservation of the energy,

$$\mathbf{M}\ddot{\mathbf{q}}_{n+1/2} + \left[\frac{\partial \Phi_i}{\partial \mathbf{q}_{n+\beta_i}}\right]^{\mathrm{T}} \left(\alpha\overline{\mathbf{\Phi}}_{n+1/2} + \boldsymbol{\lambda}^{*}\right) = \mathbf{Q}_c + (\mathbf{Q}_{nc})_{n+1/2} \tag{7}$$

$$\mathbf{q}_{n+1} = \mathbf{q}_n + h\dot{\mathbf{q}}_{n+1/2} \tag{8}$$

$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + h\ddot{\mathbf{q}}_{n+1/2} \tag{9}$$

$$\boldsymbol{\lambda}_{n+1}^{*(i+1)} = \boldsymbol{\lambda}_{n+1}^{*(i)} + \alpha\overline{\mathbf{\Phi}}_{n+1/2} \tag{10}$$

In (7), $\mathbf{Q}_c$, and $(\mathbf{Q}_{nc})$ are the contributions of the conservative and non-conservative forces to the generalized forces vector, $\overline{(\cdot)}_{n+1/2} = [(\cdot)_{n+1} + (\cdot)_n]/2$ and $(\cdot)_{n+\beta_i}$ denotes evaluation at $\mathbf{q}_{n+\beta_i}$, which can be calculated as $\mathbf{q}_{n+\beta_i} = \mathbf{q}_n + \beta_i(\mathbf{q}_{n+1} - \mathbf{q}_n)$. All the remaining terms have the same meaning already explained in the previous section. In (8) and (9), h is the time-step. To guarantee the conservative behavior of the constraint forces, the parameter $\beta_i \in [0, 1]$ has to be computed for each constraint and at each time-step, by imposing the following equality,

$$\left[\frac{\partial \Phi_i}{\partial \mathbf{q}_{n+\beta_i}}\right]^{\mathrm{T}} (\mathbf{q}_{n+1} - \mathbf{q}_n) = \mathbf{\Phi}_{n+1} - \mathbf{\Phi}_n \tag{11}$$

The form of the term $\mathbf{Q}_c$ depends on the particular expression of each conservative force, so it is case-dependent and will not be described here.

The proposed algorithm given by (7), (8), (9) and (10) achieves exact conservation of the total energy in conservative systems, and exact fulfillment of the position constraints.

The formulation proposed in this section has proven to stabilize the behavior of the original formulation (2), but using a different strategy than the formulation proposed in Section III: the control of the energy stored in the conservative forces (including the constraint forces).

Moreover, it is possible to use the projection strategy described in Section III, together with the conserving formulation described here, in order to enforce the constraints at velocity and acceleration levels.

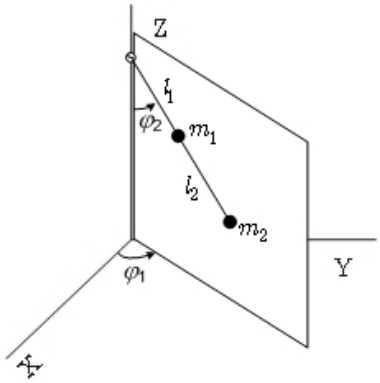

Fig. 1. Numerical simulation: an spherical compound pendulum.

V. Numerical Simulations I: Spherical compound pendulum

The first problem analyzed is a spherical compound pendulum, Goicolea and Garca Orden (2002), (see Fig.1). The system is composed of two particles with masses $m_1 = m_2 = 1kg$, placed respectively in the center and end of a massless rod of total length $l_1 + l_2$ with $l_1 = l_2 = 1m$.The pendulum is released from the position $\varphi_1 = 0$, $\varphi_2 = \pi/2$, with initial velocities $\dot{\varphi_1} = 0.5$, $\dot{\varphi_2} = 0$ The system is modeled in natural coordinates.

A. Stability problems of the index-3 augmented Lagrangian with standard integrators

The simulation is carried out for $5\,s.$, based on the formulation (2), and integrated with the trapezoidal rule and a time-step of $25\,ms$. The penalty factor chosen is 10^7.

Fig.2 and Fig.3 show the time history of the norm of the constraints at velocity level, $\left\|\dot{\mathbf{\Phi}}\right\|$, and the total energy respectively. The drift-off effect of the velocity constraints can be appreciated in Fig.2. In Fig.3, it is observed that the total energy of the system is significantly affected, and grows in an uncontrolled manner.

B. Stabilized formulations: coordinate projection vs. conserving formulation

The formulations of Sections III and IV are analyzed here. Those formulations were designed to overcome the problems shown by the augmented Lagrangian formulation, explained in the Section V-A.

Fig.4 and Fig.5 show, as expected, that the scheme with coordinate projections fulfills better the constraints at velocity and acceleration levels. Fig.6 shows that the conserving scheme achieves the exact conservation of the total energy.

The important point is that both schemes provide an adequate stabilization of the equations (2).

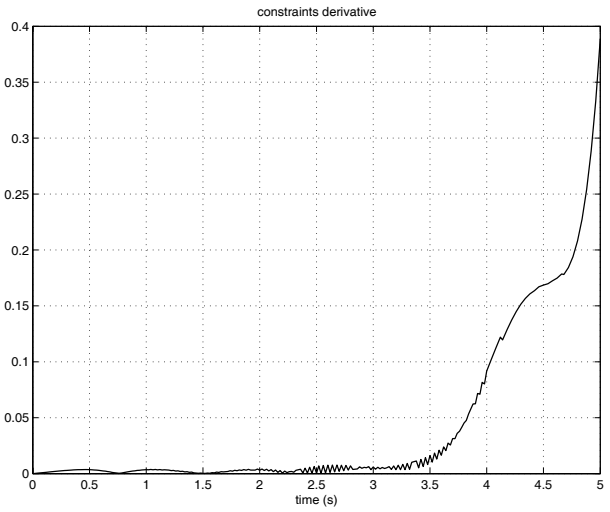

Fig. 2. Constraints derivative behavior.

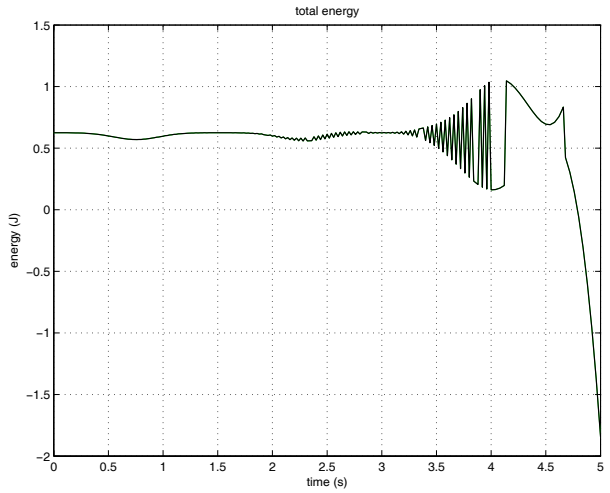

Fig. 3. Total energy behavior.

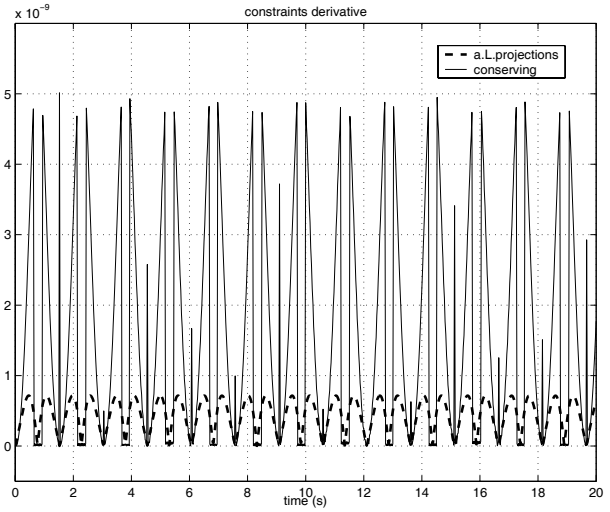

Fig. 4. Constraints derivative behavior.

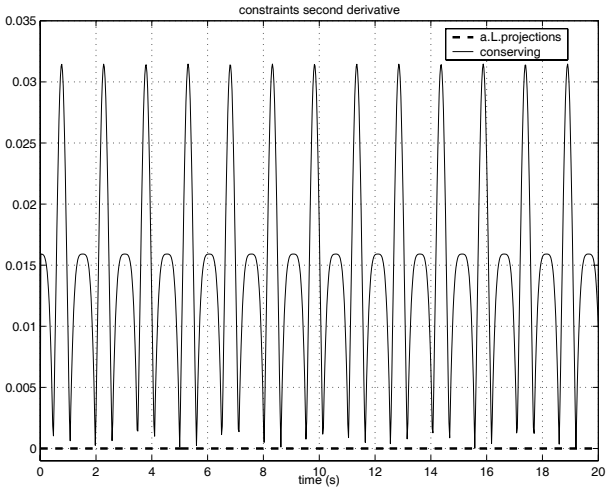

Fig. 5. Constraints second derivative behavior.

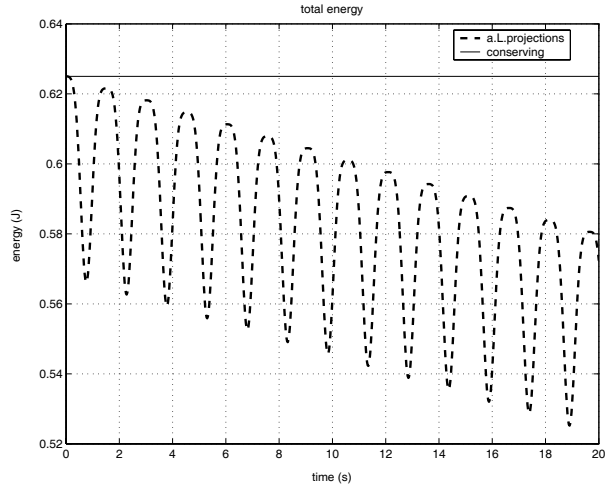

Fig. 6. Total energy behavior.

VI. Numerical Simulations II: The Iltis vehicle

As a good complex and realistic example to test methods for demanding real-time multibody applications, the full model of the Iltis vehicle [15], illustrated in Fig.7, and used as a benchmark problem to check multibody dynamic codes, has been chosen. The simulation which has served to compare the different methods, consists of $8\,s.$ of motion with the vehicle going up an inclined ramp and then down a series of stairs, starting at a horizontal speed of $5\,m/s$ (the road profile is shown in Fig.8). A rather violent motion is undergone by the vehicle, reaching acceleration peaks of up to $5\,g$.

A. Coordinate projection vs. conserving formulation

Several analyses were performed with the formulations of Section III and Section IV, using different time-steps. Fig.9 shows the response of both formulations with a time-

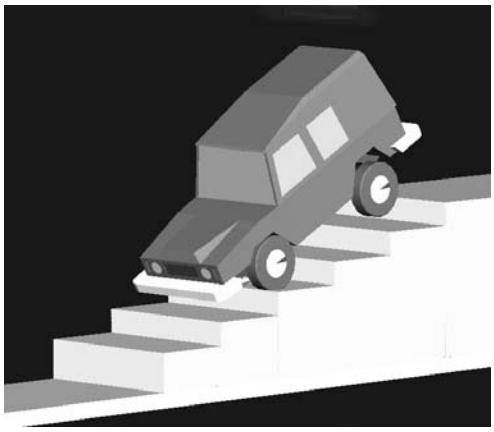

Fig. 7. Iltis vehicle.

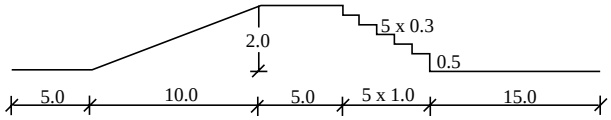

Fig. 8. Road profile.

step of $10\,ms$.

The augmented Lagrangian formulation with projections (Section III), used with a standard integrator, the trapezoidal rule, and is capable of solving this violent maneuver providing a good solution (Fig.9).

On the other hand, the conserving formulation (Section IV) fails to solve the same maneuver with any time-step. The method is not stable enough to achieve convergence when the maneuver becomes violent, and it cannot pass through the first impact of the front wheels with the stairs, so that it only provides a solution for the first $4\,s$. of the simulation (Fig.9).

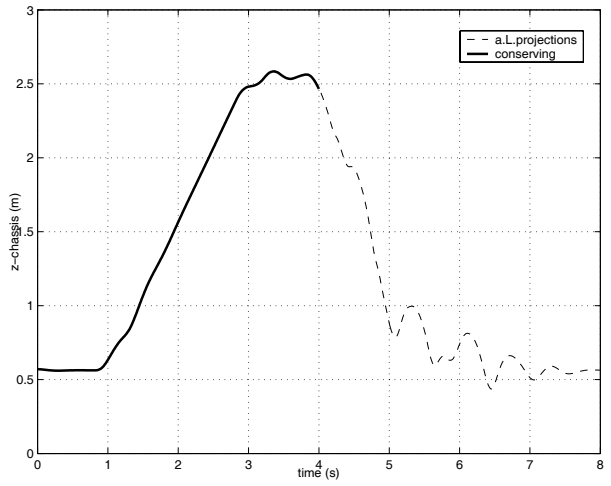

Fig. 9. Projections and trapezoidal rule vs. conserving. $h = 0.01s$

B. Combination of both stabilizing strategies

It was noted in Section VI-B that the conserving formulation cannot integrate the motion of the Iltis vehicle with the road profile shown in Fig.8.

Moreover, it was pointed out in Section IV, that the projection strategy is compatible with the conserving formulation. The resulting scheme brings together two different ways of stabilizing the augmented Lagrangian equations (2): keeping the energy of the system bounded, and keeping the solution onto the constraints manifold.

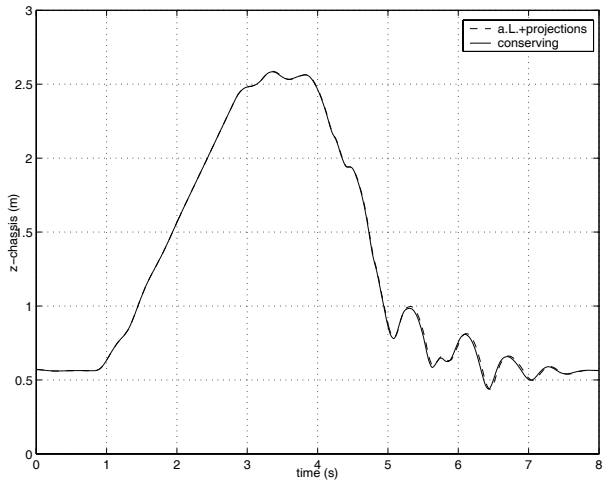

Fig. 10. Projections and trapezoidal rule vs. projections + conserving. $h = 0.01s$

With this combined scheme of conserving formulation and projections, it is possible to solve the motion of the Iltis vehicle with a higher precision than that given by the trapezoidal rule with projections, for the same time-step.

Fig.10 shows that the trapezoidal rule prolongs the period of the solution in comparison whit the conserving formulation. This effect is even more apparent with bigger time-steps. This circumstance make possible to get acceptable solutions for higher time-steps, if the conserving augmented Lagrangian scheme with projections is used.

VII. Conclusions

Two different methods to integrate the equations of constrained multibody systems have been described, both of them based on an augmented Lagrangian formulation. The formulations described, use two different strategies to stabilize the numerical behavior: one strategy is based on the projection of velocities and accelerations onto the constraints manifolds, and the other one on a specialized integrator which exactly conserves the energy for conservative systems. The conserving formulation shows a very good behavior, especially in long term simulations of conservative systems, for which the conservation is very important. The trapezoidal rule with projections shows a very robust

behavior along with an acceptable precision. The formulation is not as adequate as the previous one for long term simulations of conservative systems, but it is a good candidate for demanding real-time simulations, due to its high robustness. The combination of the conserving scheme with the coordinate projection technique results in an algorithm which brings together the advantages of both strategies, and it can be a good candidate for real-time simulations.

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