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APPLICATION CRITERIA FOR CONSERVING INTEGRATORS AND PROJECTION METHODS IN MULTIBODY DYNAMICS

Daniel Dopico*
Laboratorio de Ingenieria Mecanica
Escuela Politecnica Superior
Universidad de La Coruña
Ferrol, Spain
Email: ddopico@udc.es

Javier Cuadrado
Laboratorio de Ingenieria Mecanica
Escuela Politecnica Superior
Universidad de La Coruña
Ferrol, Spain
Email: javicuad@cdf.udc.es

Juan C. Garcia Orden
Grupo de Mecanica Computacional
Escuela Tecnica Superior de
Ingenieros de Caminos Canales y Puertos
Universidad Politecnica de Madrid
Madrid, Spain
Email: jcgarcia@mecanica.upm.es

Alberto Luaces
Laboratorio de Ingenieria Mecanica
Escuela Politecnica Superior
Universidad de La Coruña
Ferrol, Spain
Email: aluaces@udc.es

ABSTRACT

This work presents the application to the dynamics of multibody systems of two methods based on augmented Lagrangian techniques, compares them, and gives some criteria for its use in realistic problems.

The methods are an augmented Lagrangian method with orthogonal projections of velocities and accelerations, and an augmented Lagrangian energy conserving method. Both methods were presented by the authors in a very recent work, but it was not complete since the testing and the comparison of the methods was done by simulating a simple and academic example, and that was not sufficient to draw conclusions in terms of efficiency. For this work, the whole model of a vehicle has been simulated through both formulations, and their performance compared for such a large and realistic problem.

*Address all correspondence to this author.

NOMENCLATURE

$\mathbf{M} \in \Re^{nxn}$ Mass matrix.
$\mathbf{Q} \in \Re^{n}$ Vector of generalized forces.
$\lambda \in \Re^{m}$ Lagrange multipliers vector.
α Penalty factor
$\Phi \in \Re^{m}$ Constraints vector
$\Phi_{\mathbf{q}} \in \Re^{mxn}$ Jacobian matrix of the constraints vector

INTRODUCTION

Some of the most robust methods for the dynamics of multibody systems make use of natural or fully Cartesian coordinates [9], which are dependent by nature, and lead to systems of differential-algebraic equations of motion (DAE) [5].

In order to solve such DAE in natural coordinates, different formulations have been developed, like Baumgarte stabilization [1], penalty and augmented Lagrangian schemes [2], or velocity transformations [19], [17].

Formulations based on penalty and augmented Lagrangian methods have the advantages of being very simple, computationally inexpensive and very robust in the presence of singular configurations or redundant constraints [4].

Generally, it can be said that the choice of the dynamic formulation determines that of the numerical integrator. In this direction, different authors proposed several options to successfully integrate the equations arising from constrained multibody systems, using integrators coming from the field of structural dynamics [6, 9, 12].

In [4, 6], the use of augmented Lagrangian techniques with penalty only at position level along with the trapezoidal rule was proposed. In order to guarantee the correct satisfaction of the constraints, velocity and acceleration projections were proposed. More recently, in [8] it was proposed the use of augmented Lagrangian techniques with other integrators of the Generalized-α family along with projections, obtaining very good behavior. The advantages of the projections were the simplicity and that they could be used with a great variety of integrators. The projections were responsible for maintaining the stability of the formulation.

On the other hand, other authors, [10, 13, 14], developed a formulation based on an energy conserving penalty scheme, enforcing the constraints at the position level, and applied it to the dynamics of multibody systems parametrized with natural coordinates. In this case, the use of penalty at position level had the advantage of enabling to derive the constraint forces from a potential function: the constraint energy. The formulation employed an energy-momentum integrator as integration scheme [15, 18], so that the conservation of the total energy of the system was imposed by construction of the algorithm. Here, the stabilization of the penalty equations of motion arose in a natural manner from the integration scheme.

Very recently in [11] an augmented Lagrangian formulation with orthogonal projections in velocities and accelerations was presented. The projections proposed in that work are more general than the projections mentioned before. Also in [11] was presented an augmented Lagrangian energy conserving formulation, different of that presented in [10, 13, 14], which was a penalty formulation.

In this paper, the work presented in [11] is extended, since the numerical testing in the mentioned paper, was done with a very simple and academic example, while in the present paper, the whole model of a car is simulated using the same formulations presented in [11], in order to compare them in terms of efficiency.

AUGMENTED LAGRANGIAN FORMULATION

Many different methods have been proposed in the literature for the dynamics of constrained mechanical systems. The formulation of the equations of a constrained mechanical system poses some numerical difficulties. These difficulties are, in general, different for each formulation and solution method, but are typically related to stability properties of the numerical scheme. Such problems motivate the interest in developing algorithms capable of providing stable and accurate solutions.

In this work, we restrict ourselves to the methods based on the augmented Lagrangian formulation, which can be understood as a compromise between the Lagrange multiplier formulation, and the penalty formulations.

Description of the formulation

Let us consider a multibody system, with a configuration defined by a vector $\mathbf{q} \in \Re^n$. The system is also subjected to m holonomic constraints $\Phi \in \Re^m$ involving the different points and vectors of the system. The dynamic equations constitute an index-3 DAE system of $n+m$ equations given by:

$$\mathbf{M}\ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^{\mathrm{T}}\alpha\Phi + \Phi_{\mathbf{q}}^{\mathrm{T}}\lambda^* = \mathbf{Q} \qquad , \qquad \Phi = 0 \tag{1}$$

In practice, the augmented Lagrangian formulation, transforms the DAE into a system of ordinary differential equations (ODE), defining an iterative update for the multipliers, given by $\lambda^*_{i+1} = \lambda^*_i + \alpha\Phi$ verifiying that $\lambda^* \to \lambda$ as $i \to \infty$, which means that, in the limit, the iterative scheme for the Lagrange multipliers leads to the true Lagrange multipliers. Moreover, the iteration update for the multipliers, prevents introducing the Lagrange multipliers as unknowns of the problem, so that the system of $n+m$ equations is replaced by another one of size only n.

$$\mathbf{M}\ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^{\mathrm{T}}\alpha\Phi + \Phi_{\mathbf{q}}^{\mathrm{T}}\lambda^* = \mathbf{Q} \quad , \quad \lambda^*_{i+1} = \lambda^*_i + \alpha\Phi \tag{2}$$

In order to solve the nonlinear equations (2), the use of Newton-Raphson schemes is recommended, and then it is possible and convenient to mix up the Newton-Raphson iteration and the Lagrange multipliers iteration. Several authors have used methods based on the equations (2) for the dynamics of multibody systems. This equations show a slightly unstable behavior with most of the commonly used integrators, characterized by the drift-off effect and a progressive and unbounded growth of the total energy of the system.

FORMULATION A: INDEX-3 AUGMENTED LAGRANGIAN WITH PROJECTIONS

This formulation was presented in [11]. It is based on the augmented Lagrangian equations (2).

The augmented Lagrangian formulation can be combined with any standard integrator, and achieves the exact fulfillment of the position constraints, but usually exhibits an unstable behavior, even with ODE integrators suited to stiff systems.

Based on previous results obtained in the literature, it can be said that the numerical solution of a constrained mechanical system seems to be more stable on the constraints manifold than off it. Based on this fact, the exact enforcement of the constraints at velocity and acceleration levels ($\dot{\Phi} = \mathbf{0}$ and $\ddot{\Phi} = \mathbf{0}$ respectively), which is not accomplished by the augmented Lagrangian formulation (2), is expected to stabilize the numerical solution.

The way to enforce the constraints proposed in this section is the so-called coordinate projection technique. The velocities and accelerations coming from the integrator, are projected onto the constraints manifolds $\dot{\Phi} = \mathbf{0}$ and $\ddot{\Phi} = \mathbf{0}$ respectively, by means of the following expressions:

$$(\mathbf{P} + \Phi_{\mathbf{q}}^{\mathrm{T}} k\alpha\Phi_{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{P}\dot{\mathbf{q}}^{*} - \Phi_{\mathbf{q}}^{\mathrm{T}} k\alpha\Phi_{t} \tag{3}$$

$$(\mathbf{P} + \Phi_{\mathbf{q}}^{\mathrm{T}} k\alpha\Phi_{\mathbf{q}})\ddot{\mathbf{q}} = \mathbf{P}\ddot{\mathbf{q}}^{*} - \Phi_{\mathbf{q}}^{\mathrm{T}} k\alpha(\dot{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} + \dot{\Phi}_{t}) \tag{4}$$

where $\mathbf{P}$ is the projection matrix, $\dot{\mathbf{q}}^{*}$ and $\ddot{\mathbf{q}}^{*}$ are the velocities and accelerations coming from the integrator, $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ are the projected velocities and accelerations, and k is a real constant.

The choice of the projection matrix $\mathbf{P}$ is of key importance, since it determines the numerical stability of the solution. Moreover, a correct choice of this matrix can guarantee the unconditional energy-dissipative character of the projections, which proves to be very efficient for stabilizing the system.

By a careful choice of the projection matrix $\mathbf{P}$ and the real constant k, it is also possible to equal the LHS of the linear systems (3) and (4), $(\mathbf{P} + \Phi_{\mathbf{q}}^{\mathrm{T}} k\alpha\Phi_{\mathbf{q}})$, to the tangent matrix of the Newton-Raphson iteration for the nonlinear system (2), matrix which must be factorized only once, thus saving much time at the projection stage.

FORMULATION B: CONSERVING AUGMENTED LAGRANGIAN FORMULATION

The approach described in this section, is the augmented Lagrangian energy conserving formulation presented in [11]. The mentioned formulation, is based on the equations (2), repeated here for clarity:

$$\mathbf{M}\ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^{\mathrm{T}}\alpha\Phi + \Phi_{\mathbf{q}}^{\mathrm{T}}\lambda^{*} = \mathbf{Q} \quad , \quad \lambda_{i+1}^{*} = \lambda_{i}^{*} + \alpha\Phi \tag{5}$$

The key point of the conserving approach is the formulation of the constraint and conservative forces in such a way that guarantees the algorithmic conservation of the energy,

$$\mathbf{M}\ddot{\mathbf{q}}_{n+1/2} + \left[\frac{\partial\Phi_{i}}{\partial\mathbf{q}_{n+\beta_{i}}}\right]^{\mathrm{T}} \left(\alpha\overline{\Phi}_{n+1/2} + \lambda^{*}\right) = \mathbf{Q}_{c} + (\mathbf{Q}_{nc})_{n+1/2} \tag{6}$$

$$\mathbf{q}_{n+1} = \mathbf{q}_{n} + h\dot{\mathbf{q}}_{n+1/2} \tag{7}$$

$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_{n} + h\ddot{\mathbf{q}}_{n+1/2} \tag{8}$$

$$\lambda_{n+1}^{*(i+1)} = \lambda_{n+1}^{*(i)} + \alpha\overline{\Phi}_{n+1/2} \tag{9}$$

In (6), $\mathbf{Q}_{c}$, and $(\mathbf{Q}_{nc})$ are the contributions of the conservative and non-conservative forces to the generalized forces vector, $\overline{(\cdot)}_{n+1/2} = [(\cdot)_{n+1} + (\cdot)_{n}]/2$ and $(\cdot)_{n+\beta_{i}}$ denotes evaluation at $\mathbf{q}_{n+\beta_{i}}$, which can be calculated as $\mathbf{q}_{n+\beta_{i}} = \mathbf{q}_{n} + \beta_{i}(\mathbf{q}_{n+1} - \mathbf{q}_{n})$. All the remaining terms have the same meaning already explained in the previous section. In (7) and (8), h is the time-step. To guarantee the conservative behavior of the constraint forces, the parameter $\beta_{i} \in [0,1]$ has to be computed for each constraint and at each time-step, by imposing the following equality,

$$\left[\frac{\partial\Phi_{i}}{\partial\mathbf{q}_{n+\beta_{i}}}\right]^{\mathrm{T}} (\mathbf{q}_{n+1} - \mathbf{q}_{n}) = \Phi_{n+1} - \Phi_{n} \tag{10}$$

The form of the term $\mathbf{Q}_{c}$ depends on the particular expression of each conservative force, so it is case-dependent and will not be described here.

The proposed algorithm given by (6), (7), (8) and (9) achieves exact conservation of the total energy in conservative systems, and exact fulfillment of the position constraints.

The formulation B proposed in this section has proven to stabilize the behavior of the original formulation (2), but using a different strategy than the formulation A: the control of the energy stored in the conservative forces (including the constraint forces).

Moreover, it is possible to use the projection strategy described with the Formulation A, together with the conserving formulation described here, in order to enforce the constraints at velocity and acceleration levels.

NUMERICAL SIMULATIONS I: SPHERICAL COMPOUND PENDULUM

The first problem analyzed is a spherical compound pendulum, Goicolea and Garca Orden (2002), (see Figure 1). The system is composed of two particles with masses $m_{1} = m_{2} = 1kg$, placed respectively in the center and end of a massless rod of total length $l_{1} + l_{2}$ with $l_{1} = l_{2} = 1m$. The pendulum is released from the position $\varphi_{1} = 0$, $\varphi_{2} = \pi/2$, with initial velocities $\dot{\varphi}_{1} = 0.5$, $\dot{\varphi}_{2} = 0$ The system is modeled in natural coordinates.

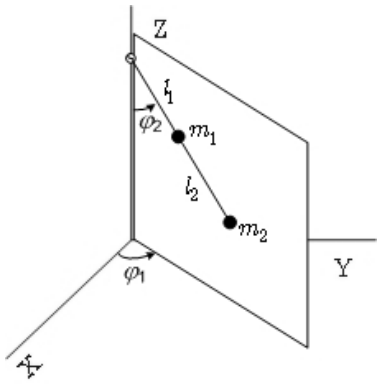

Figure 1. NUMERICAL SIMULATION: AN SPHERICAL COMPOUND PENDULUM.

Stability problems of the index-3 augmented Lagrangian with standard integrators

The simulation is carried out for $5\,s.$, based on the formulation (2), and integrated with the trapezoidal rule and a time-step of $25\,ms$. The penalty factor chosen is 10^7.

Figure 2 and Figure 3 show the time history of the norm of the constraints at velocity level, $\left\|\dot{\Phi}\right\|$, and the total energy respectively. The drift-off effect of the velocity constraints can be appreciated in Figure 2. In Figure 3, it is observed that the total energy of the system is significantly affected, and grows in an uncontrolled manner.

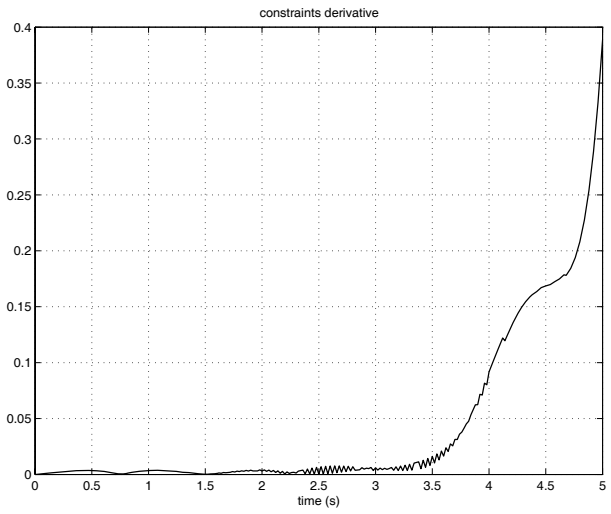

Figure 2. CONSTRAINTS DERIVATIVE BEHAVIOR.

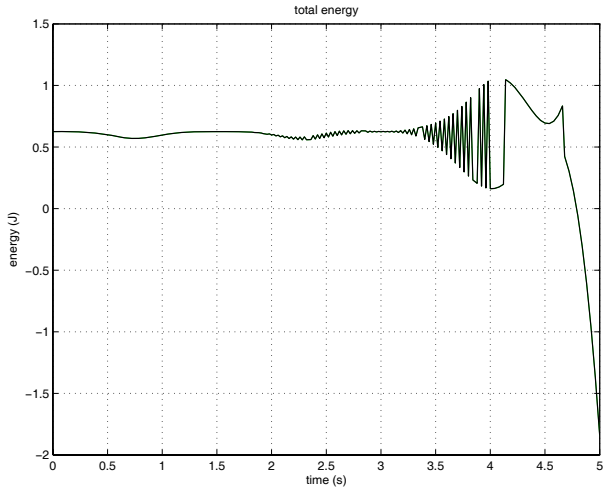

Figure 3. TOTAL ENERGY BEHAVIOR.

Stabilized formulations: Formulation A vs. Formulation B

The Formulation A and Formulation B described in previous sections are analyzed here. Those formulations were designed to overcome the problems shown by the original augmented Lagrangian formulation.

Figure 4 and Figure 5 show, as expected, that the scheme with coordinate projections fulfills better the constraints at velocity and acceleration levels. Figure 6 shows that the conserving scheme achieves the exact conservation of the total energy.

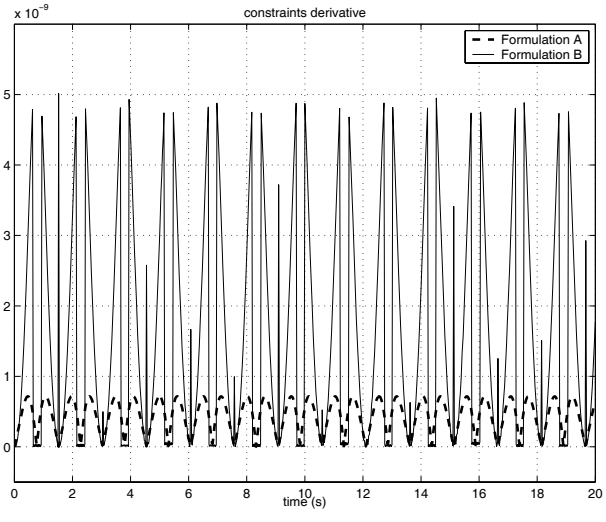

Figure 4. CONSTRAINTS DERIVATIVE BEHAVIOR.

The important point is that both schemes provide an ade-

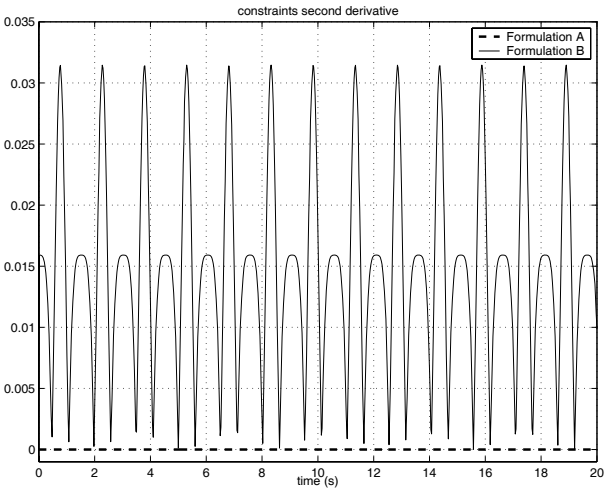

Figure 5. CONSTRAINTS SECOND DERIVATIVE BEHAVIOR.

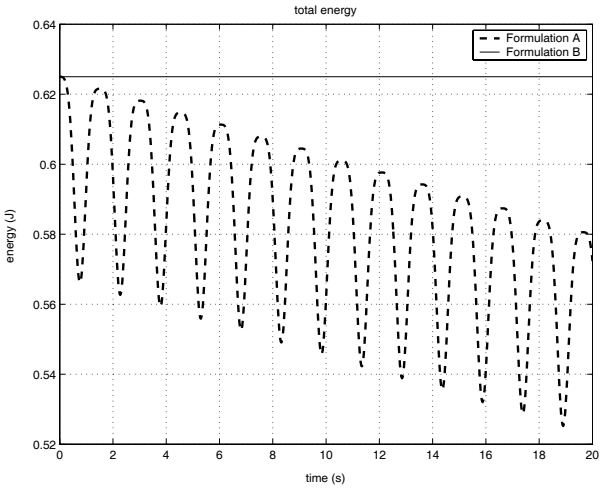

Figure 6. TOTAL ENERGY BEHAVIOR.

quate stabilization of the equations (2).

NUMERICAL SIMULATIONS II: THE ILTIS VEHICLE

As a good complex and realistic example to test methods for demanding multibody applications, the full model of the Iltis vehicle [16], illustrated in Figure 7, and used as a benchmark problem to check multibody dynamic codes, has been chosen.

Road profile I: Smooth profile

The first simulation used to compare the Formulation A and the Formulation B, consists of $5\,s$. of manoeuvre with the vehicle crossing two bumps. The first bump affects only to the right wheels and the second only to the left wheels (the road profile

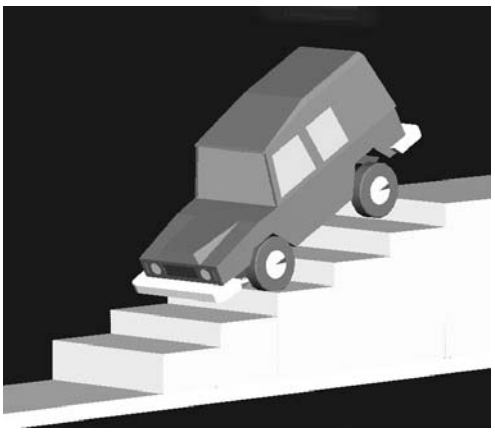

Figure 7. ILTIS VEHICLE.

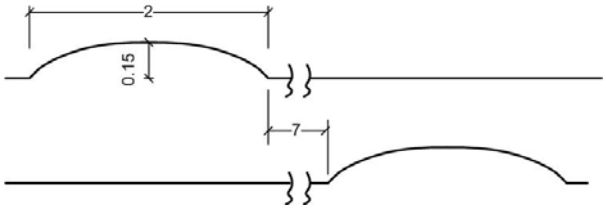

Figure 8. ROAD PROFILE I: RIGHT WHEELS (TOP) AND LEFT WHEELS (BOTTOM).

for the right and left wheels is shown in the Figure 8), starting at a horizontal speed of $5\,m/s$.

Several analyses were performed using the Formulation A and the Formulation B, with different time-steps. The Figure 9 shows the time history of the z coordinate of the chassis for both formulations using a constant time-step of $10\,ms$. The Figure 10 shows the time history of the center of the left and right wheels. The response is practically the same for both formulations. A detailed analysis of the results, shows that the Formulation A damps out part of the response and the crests are not so pronounced like in the case of the Formulation B, but this effect is barely perceptible in the figures.

The Formulation A can achieve time-steps up to $60\,ms$, while the Formulation B fails to converge for time-steps bigger than $10\,ms$. The Formulation B has more problems when the integration conditions become hard.

In the Table 1 and Table 2 the CPU-time of each formulation is presented, to solve the $5\,s$ of manoeuvre with a constant time step of $10\,ms$ (Table 1) and with the maximum time-step possible (Table 2). Even when the same time-step is chosen, the Formulation A is faster. This fact is motivated because of the higher number of iterations necessary for the Formulation B to converge, and also because of the non-symmetric tangent matrix

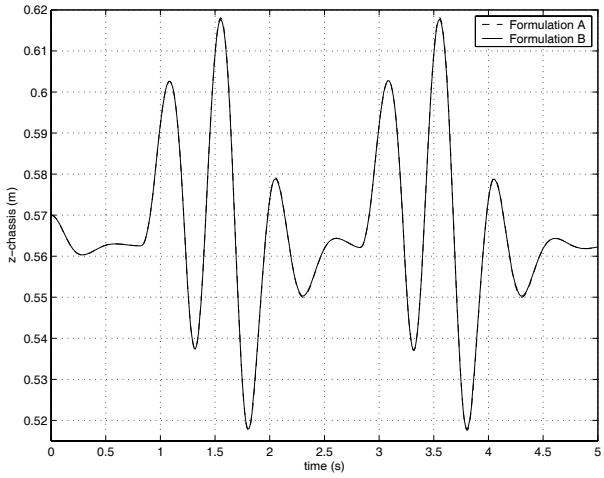

Figure 9. PROFILE I: FORMULATION A VS. FORMULATION B, Z-CHASSIS. $h = 0.01s$

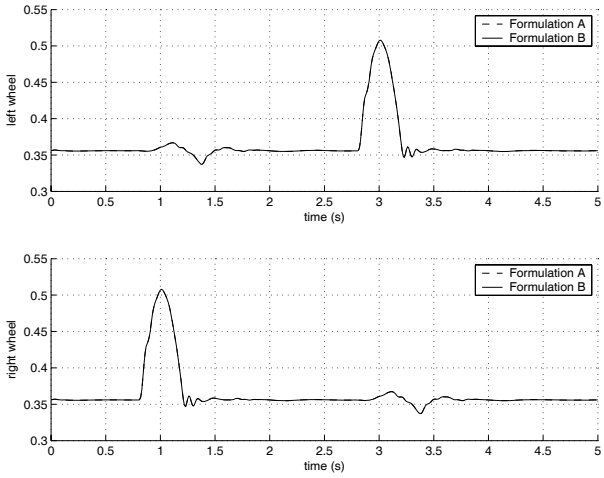

Figure 10. PROFILE I: FORMULATION A VS. FORMULATION B, WHEELS. $h = 0.01s$

which comes from the solution of the nonlinear system of equations (7).

Road profile II: Rough profile

The second simulation which has served to compare the Formulation A and the Formulation B, consists of $8\,s.$ of motion with the vehicle going up an inclined ramp and then down a series of stairs, starting at a horizontal speed of $5\,m/s$ (the road profile is shown in Figure 11). A rather violent motion is undergone by the vehicle, reaching acceleration peaks of up to $5\,g$.

Several analyses were performed with the Formulation A and the Formulation B, using different time-steps. Figure 12 shows the response of both formulations with a time-step of

Table 1. FORMULATION A VS. FORMULATION B: CPU-TIME. $h = 0.01s$

Formulation	Iterations	CPU-time (s)
A	1039	2.35
B	1485	4.40

Table 2. FORMULATION A VS. FORMULATION B: CPU-TIME. MAXIMUM TIME-STEP

Formulation	Iterations	CPU-time (s)
A	501	0.96
B	1485	4.40

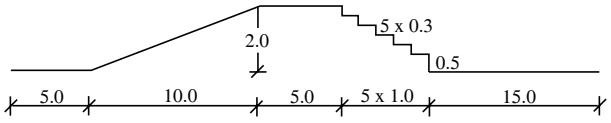

Figure 11. ROAD PROFILE II.

$10\,ms.$

The Formulation A, used with a standard integrator, the trapezoidal rule, is capable of solving this violent maneuver and provides a good solution (Figure 12).

On the other hand, the Formulation B fails to solve the same maneuver with any time-step. The method is not robust enough to achieve the convergence when the maneuver becomes violent, and it cannot pass through the first impact of the front wheels with the stairs, so that it only provides a solution for the first $4\,s.$ of the simulation (Figure 12).

Combination of both stabilizing strategies It was noted in the previous subsection that the Formulation B cannot integrate the motion of the Iltis vehicle with the road profile shown in Figure 11.

Moreover, it was pointed out previously, that the projection strategy is also compatible with the conserving Formulation B. The resulting scheme brings together two different ways of stabilizing the augmented Lagrangian equations (2): keeping the energy of the system bounded, and keeping the solution onto the constraints manifold.

With this combined scheme of conserving formulation and projections, it is possible to solve the motion of the Iltis vehicle

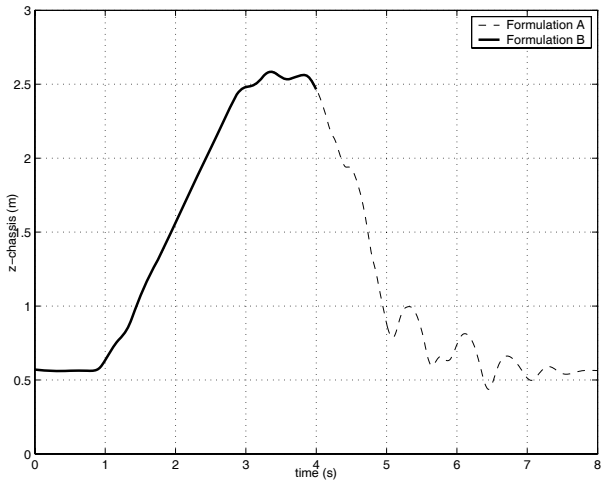

Figure 12. PROJECTIONS AND TRAPEZOIDAL RULE VS. CONSERVING. $h = 0.01s$

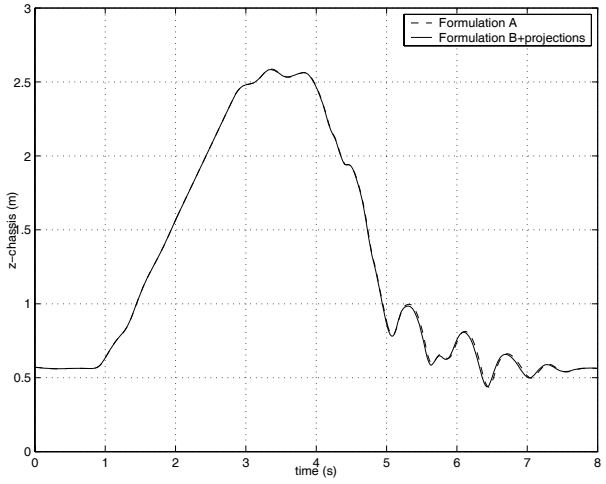

Figure 13. PROJECTIONS + TRAPEZOIDAL RULE VS. PROJECTIONS + CONSERVING. $h = 0.01s$

with a higher precision than that given by the trapezoidal rule with projections, for the same time-step.

Figure 13 shows that the trapezoidal rule prolongs the period of the solution in comparison whit the conserving formulation. This effect is even more apparent with bigger time-steps. This circumstance make possible to get acceptable solutions for higher time-steps, if the conserving augmented Lagrangian scheme with projections is used.

CONCLUSIONS

Two different methods to integrate the equations of constrained multibody systems have been described, both of them based on an augmented Lagrangian formulation.

The formulations described, use two different strategies to stabilize the numerical behavior: one strategy is based on the projection of velocities and accelerations onto the constraints manifolds, and the other one on a specialized integrator which exactly conserves the energy for conservative systems.

The simulation of the whole model of a vehicle was carried out using the formulations described in the paper. This is a very realistic and demanding simulation that permits to draw conclusions about the formulations.

The Formulation B (conserving) shows a very good behavior, especially in long term simulations of conservative systems, for which the conservation is very important, but It is clear from the experiments that this formulation does not achieve the high robustness of its opponent (Formulation A).

The Formulation A (projections) shows a very robust behavior along with an acceptable precision. The formulation is not as adequate as the previous one for long term simulations of conservative systems, but it is the best option for demanding simulations, due to its high robustness.

The combination of the conserving scheme with the coordinate projection technique results in an algorithm which brings together the advantages of both strategies.

The criteria of application for the formulations described can be the following: For very demanding and hard simulations in which the exact conservation of energy is not important, the Formulation A is better option. For smoother simulations, or long term simulations of conservative systems in which the precision is important, the Formulation B is better option.

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