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COMPUTING SENSITIVITY ANALYSIS OF VEHICLE DYNAMICS BASED ON MULTIBODY MODELS

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ABSTRACT

Vehicle dynamics simulation based on multibody dynamics techniques has become a powerful tool for vehicle systems analysis and design. As this approach evolves, more and more details are required to increase the accuracy of the simulations, to improve their efficiency, or to provide more information that will allow various types of analyses. One very important direction is the optimization of multibody systems. Sensitivity analysis of the dynamics of multibody systems is essential for design optimization. Dynamic sensitivities, when needed, are often calculated by means of finite differences but, depending of the number of parameters involved, this procedure can be very demanding in terms of time and the accuracy obtained can be very poor in many cases if real perturbations are used. In this paper, several ways to perform the sensitivity analysis of multibody systems are explored including the direct sensitivity approaches and the adjoint sensitivity ones. Finally, the techniques proposed are applied to the dynamical optimization of a five bar mechanism and a vehicle suspension system.

INTRODUCTION

Multibody dynamics has become an essential tool for vehicle systems analysis and design. The evolution during the last decades leaded to complex multibody vehicle models that consider phenomena difficult to take into account years ago and impossible to achieve with analytical models. One interesting application of the state-of-the-art multibody models of vehicles, is the design optimization of certain parts of the vehicle. Sensitivity analysis of the dynamics of multibody systems is essential for design optimization.

In general, the multibody dynamics equations, constitute an index-3 differential algebraic system of equations (DAE) that it is not usually directly solved because of the numerical difficulties involved [1, 2]. Some of the most advanced families of formulations used nowadays are based on some ideas presented in the eighties and nineties. One of this families comprise penalty and augmented Lagrangian formulations introduced in [3, 4].

The sensitivity equations for the mentioned penalty and augmented Lagrangian formulations were developed by [5] for the direct sensitivity calculation. In this paper the approach is extended to compute sensitivities using the adjoint variable theory [6]. The validity of the theoretical results introduced in the paper to calculate the sensitivities is checked comparing the di-

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rect and the adjoint approaches, comparing with numerical results and comparing with third party libraries for sensitivity analysis.

In this paper, the theory of sensitivity analysis developed is used along with the constrained optimization theory to solve the optimization of a five bar mechanism and the optimization of a vehicle suspension.

DESIGN OPTIMIZATION OF MECHANICAL SYSTEMS

The design optimization of a mechanical system is usually defined by a set of design parameters $\boldsymbol{\rho} \in \mathbb{R}^p$. These parameters are related to the geometry, materials and other characteristics that the engineer has to decide. The optimization theory can considerably help the engineer to make that decisions.

The objective of the optimization is to find a design that makes the behavior of the system optimal. The behavior of the system has to be represented mathematically by a cost or objective function $\psi = \psi(\boldsymbol{\rho})$.

Nevertheless, when the optimization is based on the dynamical behavior of the system under given inputs and initial conditions, the objective function more likely depends on the states of the system in the form $\psi = \psi(\mathbf{y})$ and the states depend on the parameters $\mathbf{y} = \mathbf{y}(\boldsymbol{\rho})$ by means of the dynamics of the system.

It is also quite usual that the vector of design variables cannot have any value and it is subjected to some design constraints $\boldsymbol{\Psi} = \boldsymbol{\Psi}(\boldsymbol{\rho})$.

Many of the most advanced methods to solve the optimization problem, require the sensitivity analysis of the objective function/constraints with respect to the parameters to get the solution of the problem. How to perform this analysis is the topic of the subsequent sections.

DESCRIPTION OF THE MULTIBODY FORMULATION

The penalty formulation presented in [3], has the following expression.

$$\left(\mathbf{M}+\boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\alpha\boldsymbol{\Phi}_{\mathbf{q}}\right)\ddot{\mathbf{q}} = \mathbf{Q}-\boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\alpha\left(\dot{\boldsymbol{\Phi}}_{\mathbf{q}}\dot{\mathbf{q}}+\dot{\boldsymbol{\Phi}}_t+2\xi\omega\dot{\boldsymbol{\Phi}}+\omega^2\boldsymbol{\Phi}\right) \quad (1)$$

In the previous expressions, $\mathbf{M} \in \mathbb{R}^{nxn}$ is the mass matrix, $\mathbf{Q} \in \mathbb{R}^{\mathbf{n}}$ is the generalized forces vector, $\boldsymbol{\Phi}$ is the constraints vector, $\boldsymbol{\Phi}_{\mathbf{q}} = \dfrac{\partial \boldsymbol{\Phi}}{\partial \mathbf{q}}$ is the Jacobian matrix of the constraints vector, $\boldsymbol{\Phi}_t = \dfrac{\partial \boldsymbol{\Phi}}{\partial t}$, α is the penalty factor and ξ, ω are coefficients of the method. As usual, the upper dot stands for the total temporal derivative.

Let's define the following:

$$\bar{\mathbf{M}} = \mathbf{M}+\boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\alpha\boldsymbol{\Phi}_{\mathbf{q}} \quad (2)$$

$$\bar{\mathbf{Q}} = \mathbf{Q}-\boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}\alpha\left(\dot{\boldsymbol{\Phi}}_{\mathbf{q}}\dot{\mathbf{q}}+\dot{\boldsymbol{\Phi}}_t+2\xi\omega\dot{\boldsymbol{\Phi}}+\omega^2\boldsymbol{\Phi}\right) \quad (3)$$

Replacing (2) and (3) in (1), the following second order ODE-like system in dependent coordinates is obtained.

$$\bar{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{q}} = \bar{\mathbf{Q}}(t,\mathbf{q},\dot{\mathbf{q}}) \quad (4)$$

Let's suppose a multibody system described by the equations of motion (4) and dependent on some design parameters $\boldsymbol{\rho} \in \mathbb{R}^p$ (typically masses, lengths, or other parameters related to forces chosen by the engineer). Then the equations of motion become.

$$\bar{\mathbf{M}}(\mathbf{q},\boldsymbol{\rho})\ddot{\mathbf{q}} = \bar{\mathbf{Q}}(t,\mathbf{q},\dot{\mathbf{q}},\boldsymbol{\rho}) \quad (5)$$

Note that the mass matrix and generalized forces vector in (5) are now dependent of a parameter set $\boldsymbol{\rho}$ that contains the design variables of the system and therefore $\mathbf{q} = \mathbf{q}(t,\boldsymbol{\rho})$, $\dot{\mathbf{q}} = \dot{\mathbf{q}}(t,\boldsymbol{\rho})$, $\ddot{\mathbf{q}} = \ddot{\mathbf{q}}(t,\boldsymbol{\rho})$ also, being t the time variable.

DIRECT SENSITIVITY APPROACH

The direct approach for the sensitivity analysis using the formulation of section was initially developed in [5]. The sensitivity analysis involves obtaining the sensitivity of a cost function defined in terms of some states and design parameters of the system. The objective functions considered here will have the following form.

$$\psi = \int_{t_0}^{t_F} g(\mathbf{q},\dot{\mathbf{q}},\boldsymbol{\rho})\,\mathrm{dt} \quad (6)$$

The gradient of the cost function (6) can be obtained by the following expression.

$$\nabla_{\boldsymbol{\rho}}\psi^{\mathrm{T}} = \frac{\mathrm{d}\psi}{\mathrm{d}\boldsymbol{\rho}} = \int_{t_0}^{t_F}\left(\frac{\partial g}{\partial \mathbf{q}}\frac{\partial \mathbf{q}}{\partial \boldsymbol{\rho}}+\frac{\partial g}{\partial \dot{\mathbf{q}}}\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\rho}}+\frac{\partial g}{\partial \boldsymbol{\rho}}\right)\mathrm{dt} \quad (7)$$

Using the common notation of a sub-index to express partial derivatives and commuting the temporal and parameter derivatives:

$$\frac{\partial \mathbf{q}}{\partial \boldsymbol{\rho}} = \mathbf{q}_{\boldsymbol{\rho}} \quad (8)$$

$$\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\rho}} = \frac{d}{dt}\frac{\partial \mathbf{q}}{\partial \boldsymbol{\rho}} = \dot{\mathbf{q}}_{\boldsymbol{\rho}} \quad (9)$$

Then equation (7) becomes:

$$\nabla_{\boldsymbol{\rho}} \psi = \int_{t_0}^{t_F} \left(\frac{\partial g}{\partial \mathbf{q}} \mathbf{q}_{\boldsymbol{\rho}} + \frac{\partial g}{\partial \dot{\mathbf{q}}} \dot{\mathbf{q}}_{\boldsymbol{\rho}} + \frac{\partial g}{\partial \boldsymbol{\rho}} \right)^{\mathrm{T}} \mathrm{dt} \tag{10}$$

Where the derivatives of function g are known, since the objective function has a known expression and the derivatives $\mathbf{q}_{\boldsymbol{\rho}}$ and $\dot{\mathbf{q}}_{\boldsymbol{\rho}}$ are the sensitivities of the solution of the dynamical equations (5), that need to be obtained differentiating them like follows.

$$\frac{\mathrm{d}\bar{\mathbf{M}}}{\mathrm{d}\boldsymbol{\rho}} \ddot{\mathbf{q}} + \bar{\mathbf{M}} \frac{\partial \ddot{\mathbf{q}}}{\partial \boldsymbol{\rho}} = \frac{\mathrm{d}\bar{\mathbf{Q}}}{\mathrm{d}\boldsymbol{\rho}} \tag{11}$$

Expanding the total derivatives.

$$\frac{\partial \bar{\mathbf{M}}}{\partial \boldsymbol{\rho}} \ddot{\mathbf{q}} + \frac{\partial \bar{\mathbf{M}}}{\partial \mathbf{q}} \ddot{\mathbf{q}} \frac{\partial \mathbf{q}}{\partial \boldsymbol{\rho}} + \bar{\mathbf{M}} \frac{\partial \ddot{\mathbf{q}}}{\partial \boldsymbol{\rho}} = \frac{\partial \bar{\mathbf{Q}}}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial \boldsymbol{\rho}} + \frac{\partial \bar{\mathbf{Q}}}{\partial \dot{\mathbf{q}}} \frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\rho}} + \frac{\partial \bar{\mathbf{Q}}}{\partial \boldsymbol{\rho}} \tag{12}$$

Finally, defining $()_{\boldsymbol{\rho}} = \frac{\partial ()}{\partial \boldsymbol{\rho}}$, $()_{\mathbf{q}} = \frac{\partial ()}{\partial \mathbf{q}}$ and grouping terms, the following ODE, called Tangent Linear Model (TLM) is obtained.

$$\bar{\mathbf{M}} \ddot{\mathbf{q}}_{\boldsymbol{\rho}} + \bar{\mathbf{C}} \dot{\mathbf{q}}_{\boldsymbol{\rho}} + \left(\bar{\mathbf{K}} + \bar{\mathbf{M}}_{\mathbf{q}} \ddot{\mathbf{q}} \right) \mathbf{q}_{\boldsymbol{\rho}} = \bar{\mathbf{Q}}_{\boldsymbol{\rho}} - \bar{\mathbf{M}}_{\boldsymbol{\rho}} \ddot{\mathbf{q}} \tag{13}$$

$$\mathbf{q}_{\boldsymbol{\rho}}(t_0) = \mathbf{q}_{\boldsymbol{\rho}0} \tag{14}$$

$$\dot{\mathbf{q}}_{\boldsymbol{\rho}}(t_0) = \dot{\mathbf{q}}_{\boldsymbol{\rho}0} \tag{15}$$

In (13), $\bar{\mathbf{K}}$ and $\bar{\mathbf{C}}$ are given by expressions (16) and (17), $\bar{\mathbf{Q}}_{\boldsymbol{\rho}}$ by expression (18) and the terms $\bar{\mathbf{M}}_{\mathbf{q}} \ddot{\mathbf{q}}$ and $\bar{\mathbf{M}}_{\boldsymbol{\rho}} \ddot{\mathbf{q}}$ are derivatives of matrices times vectors, which are matrices obtained by means of expressions (19) and (20). How to obtain the initial conditions (14) and (15) is explained at the end of this section.

$$\begin{aligned} \bar{\mathbf{K}} = -\frac{\partial \bar{\mathbf{Q}}}{\partial \mathbf{q}} = \mathbf{K} + \boldsymbol{\Phi}_{\mathbf{qq}}^{\mathrm{T}} \alpha \left(\dot{\boldsymbol{\Phi}}_{\mathbf{q}} \dot{\mathbf{q}} + \dot{\boldsymbol{\Phi}}_t + 2\xi\omega \dot{\boldsymbol{\Phi}} + \omega^2 \boldsymbol{\Phi} \right) + \\ \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}} \alpha \left(\left(\dot{\boldsymbol{\Phi}}_{\mathbf{q}} \dot{\mathbf{q}} \right)_{\mathbf{q}} + \dot{\boldsymbol{\Phi}}_{t\mathbf{q}} + 2\xi\omega \left(\boldsymbol{\Phi}_{\mathbf{qq}} \dot{\mathbf{q}} + \boldsymbol{\Phi}_{t\mathbf{q}} \right) + \omega^2 \boldsymbol{\Phi}_{\mathbf{q}} \right) \end{aligned} \tag{16}$$

$$\bar{\mathbf{C}} = -\frac{\partial \bar{\mathbf{Q}}}{\partial \dot{\mathbf{q}}} = \mathbf{C} + \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}} \alpha \left(\boldsymbol{\Phi}_{\mathbf{qq}} \dot{\mathbf{q}} + \dot{\boldsymbol{\Phi}}_{\mathbf{q}} + \boldsymbol{\Phi}_{tq} + 2\xi\omega \boldsymbol{\Phi}_{\mathbf{q}} \right) \tag{17}$$

$$\begin{aligned} \bar{\mathbf{Q}}_{\boldsymbol{\rho}} = \frac{\partial \bar{\mathbf{Q}}}{\partial \boldsymbol{\rho}} = \mathbf{Q}_{\boldsymbol{\rho}} - \boldsymbol{\Phi}_{\mathbf{q}\boldsymbol{\rho}}^{\mathrm{T}} \alpha \left(\dot{\boldsymbol{\Phi}}_{\mathbf{q}} \dot{\mathbf{q}} + \dot{\boldsymbol{\Phi}}_t + 2\xi\omega \dot{\boldsymbol{\Phi}} + \omega^2 \boldsymbol{\Phi} \right) - \\ \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}} \alpha \left(\left(\dot{\boldsymbol{\Phi}}_{\mathbf{q}} \dot{\mathbf{q}} \right)_{\boldsymbol{\rho}} + \dot{\boldsymbol{\Phi}}_{t\boldsymbol{\rho}} + 2\xi\omega \dot{\boldsymbol{\Phi}}_{\boldsymbol{\rho}} + \omega^2 \boldsymbol{\Phi}_{\boldsymbol{\rho}} \right) \end{aligned} \tag{18}$$

$$\begin{aligned} \bar{\mathbf{M}}_{\mathbf{q}} \ddot{\mathbf{q}} = \frac{\partial \bar{\mathbf{M}}}{\partial \mathbf{q}} \ddot{\mathbf{q}} = \frac{\partial \left(\mathbf{M} + \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}} \alpha \boldsymbol{\Phi}_{\mathbf{q}} \right)}{\partial \mathbf{q}} \ddot{\mathbf{q}} = \\ \mathbf{M}_{\mathbf{q}} \ddot{\mathbf{q}} + \boldsymbol{\Phi}_{\mathbf{qq}}^{\mathrm{T}} \alpha \left(\boldsymbol{\Phi}_{\mathbf{q}} \ddot{\mathbf{q}} \right) + \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}} \alpha \boldsymbol{\Phi}_{\mathbf{qq}} \ddot{\mathbf{q}} \end{aligned} \tag{19}$$

$$\begin{aligned} \bar{\mathbf{M}}_{\boldsymbol{\rho}} \ddot{\mathbf{q}} = \frac{\partial \bar{\mathbf{M}}}{\partial \boldsymbol{\rho}} \dot{\mathbf{v}} = \frac{\partial \left(\mathbf{M} + \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}} \alpha \boldsymbol{\Phi}_{\mathbf{q}} \right)}{\partial \boldsymbol{\rho}} \ddot{\mathbf{q}} = \\ \mathbf{M}_{\boldsymbol{\rho}} \ddot{\mathbf{q}} + \boldsymbol{\Phi}_{\mathbf{q}\boldsymbol{\rho}}^{\mathrm{T}} \alpha \left(\boldsymbol{\Phi}_{\mathbf{q}} \ddot{\mathbf{q}} \right) + \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}} \alpha \boldsymbol{\Phi}_{\mathbf{q}\boldsymbol{\rho}} \ddot{\mathbf{q}} \end{aligned} \tag{20}$$

In equations (16), (17), $\mathbf{K} = -\frac{\partial \mathbf{Q}}{\partial \mathbf{q}}$ and $\mathbf{C} = -\frac{\partial \mathbf{Q}}{\partial \dot{\mathbf{q}}}$ respectively. For equations (16), (17), (18), (19) and (20), $\boldsymbol{\Phi}_{\mathbf{qq}}$ and $\boldsymbol{\Phi}_{\mathbf{q}\boldsymbol{\rho}}$ stand for the partial derivatives of the Jacobian matrix $\boldsymbol{\Phi}_{\mathbf{q}}$ with respect to the vectors $\mathbf{q}$ and $\boldsymbol{\rho}$. The products of the derivatives times the vectors involved have to be performed in the following way.

$$\boldsymbol{\Phi}_{\mathbf{qq}}^{\mathrm{T}} \alpha \mathbf{a} = \left[\frac{\partial \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}}{\partial q_1} \alpha \mathbf{a} \ldots \frac{\partial \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}}{\partial q_i} \alpha \mathbf{a} \ldots \frac{\partial \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}}{\partial q_n} \alpha \mathbf{a} \right] \tag{21}$$

$$\boldsymbol{\Phi}_{\mathbf{q}\boldsymbol{\rho}}^{\mathrm{T}} \alpha \mathbf{a} = \left[\frac{\partial \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}}{\partial \rho_1} \alpha \mathbf{a} \ldots \frac{\partial \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}}{\partial \rho_i} \alpha \mathbf{a} \ldots \frac{\partial \boldsymbol{\Phi}_{\mathbf{q}}^{\mathrm{T}}}{\partial \rho_p} \alpha \mathbf{a} \right] \tag{22}$$

$$\boldsymbol{\Phi}_{\mathbf{qq}} \mathbf{a} = \left[\frac{\partial \boldsymbol{\Phi}_{\mathbf{q}}}{\partial q_1} \mathbf{a} \ldots \frac{\partial \boldsymbol{\Phi}_{\mathbf{q}}}{\partial q_i} \mathbf{a} \ldots \frac{\partial \boldsymbol{\Phi}_{\mathbf{q}}}{\partial q_n} \mathbf{a} \right] \tag{23}$$

being, $\mathbf{a}$, any vector after $\boldsymbol{\Phi}_{\mathbf{qq}}^{\mathrm{T}}, \boldsymbol{\Phi}_{\mathbf{q}\boldsymbol{\rho}}^{\mathrm{T}}$ or $\boldsymbol{\Phi}_{\mathbf{qq}}$

Moreover, for expression (16), the kinematic relation $\dot{\boldsymbol{\Phi}} = \boldsymbol{\Phi}_{\mathbf{q}} \dot{\mathbf{q}} + \boldsymbol{\Phi}_t$ was employed and for expression (17) the relations $\dot{\boldsymbol{\Phi}}_{\mathbf{q}\dot{\mathbf{q}}} = \boldsymbol{\Phi}_{\mathbf{qq}}$, $\dot{\boldsymbol{\Phi}}_{t\dot{\mathbf{q}}} = \boldsymbol{\Phi}_{t\mathbf{q}}$, were used. To check the last two relations, the following differentials can be considered.

$$\begin{aligned} \delta \boldsymbol{\Phi}_{\mathbf{q}} = \boldsymbol{\Phi}_{\mathbf{qq}} \delta \mathbf{q} \Rightarrow \frac{d}{dt} \delta \boldsymbol{\Phi}_{\mathbf{q}} = \dot{\boldsymbol{\Phi}}_{\mathbf{qq}} \delta \mathbf{q} + \boldsymbol{\Phi}_{\mathbf{qq}} \delta \dot{\mathbf{q}} = \\ \delta \dot{\boldsymbol{\Phi}}_{\mathbf{q}} = \dot{\boldsymbol{\Phi}}_{\mathbf{qq}} \delta \mathbf{q} + \dot{\boldsymbol{\Phi}}_{\mathbf{q}\dot{\mathbf{q}}} \delta \dot{\mathbf{q}} \Rightarrow \dot{\boldsymbol{\Phi}}_{\mathbf{q}\dot{\mathbf{q}}} = \boldsymbol{\Phi}_{\mathbf{qq}} \end{aligned} \tag{24}$$

$$\begin{aligned} \delta \boldsymbol{\Phi}_t = \boldsymbol{\Phi}_{t\mathbf{q}} \delta \mathbf{q} \Rightarrow \frac{d}{dt} \delta \boldsymbol{\Phi}_t = \dot{\boldsymbol{\Phi}}_{t\mathbf{q}} \delta \mathbf{q} + \boldsymbol{\Phi}_{t\mathbf{q}} \delta \dot{\mathbf{q}} = \\ \delta \dot{\boldsymbol{\Phi}}_t = \dot{\boldsymbol{\Phi}}_{t\mathbf{q}} \delta \mathbf{q} + \dot{\boldsymbol{\Phi}}_{t\dot{\mathbf{q}}} \delta \dot{\mathbf{q}} \Rightarrow \dot{\boldsymbol{\Phi}}_{t\dot{\mathbf{q}}} = \boldsymbol{\Phi}_{t\mathbf{q}} \end{aligned} \tag{25}$$

For equations (19) and (20) the terms $\mathbf{M}_{\mathbf{q}} \ddot{\mathbf{q}}$ and $\mathbf{M}_{\boldsymbol{\rho}} \ddot{\mathbf{q}}$ can be calculated by means of the following expressions.

$$\mathbf{M}_{\mathbf{q}} \ddot{\mathbf{q}} = \left[\frac{\partial \mathbf{M}}{\partial q_1} \ddot{\mathbf{q}} \ldots \frac{\partial \mathbf{M}}{\partial q_i} \ddot{\mathbf{q}} \ldots \frac{\partial \mathbf{M}}{\partial q_n} \ddot{\mathbf{q}} \right] \tag{26}$$

$$\mathbf{M}_{\boldsymbol{\rho}} \ddot{\mathbf{q}} = \left[\frac{\partial \mathbf{M}}{\partial \rho_1} \ddot{\mathbf{q}} \ldots \frac{\partial \mathbf{M}}{\partial \rho_i} \ddot{\mathbf{q}} \ldots \frac{\partial \mathbf{M}}{\partial \rho_p} \ddot{\mathbf{q}} \right] \tag{27}$$

Finally, since the coordinates $\mathbf{q}$ are not independent but related by the constraint equations $\boldsymbol{\Phi} = \mathbf{0}$ and the velocities $\dot{\mathbf{q}}$ are related by the constraint derivatives $\dot{\mathbf{\Phi}} = \mathbf{\Phi_q}\dot{\mathbf{q}} + \mathbf{\Phi}_t = \mathbf{0}$, the initial conditions (14) and (15) can be obtained by means of the following calculations.

$$\frac{\mathrm{d}\mathbf{\Phi}(t_0)}{\mathrm{d}\boldsymbol{\rho}} = \mathbf{0} \rightarrow \mathbf{\Phi_q}\mathbf{q}_{\boldsymbol{\rho}0} = -\mathbf{\Phi}_{\boldsymbol{\rho}} \tag{28}$$

$$\frac{\mathrm{d}\dot{\mathbf{\Phi}}(t_0)}{\mathrm{d}\boldsymbol{\rho}} = \mathbf{0} \rightarrow \mathbf{\Phi_q}\dot{\mathbf{q}}_{\boldsymbol{\rho}0} = -\mathbf{\Phi}_{\mathbf{q}\boldsymbol{\rho}}\dot{\mathbf{q}} - \mathbf{\Phi}_{t\boldsymbol{\rho}} \tag{29}$$

Assuming that $\mathbf{\Phi_q}$ has full row rank, $n-m$ independent sensitivities can be chosen from (28) and $n-m$ independent "velocity" sensitivities from (29). That means that the impact of the parameters on the initial configuration of the system can be decided as an input to the problem.

ADJOINT SENSITIVITY APPROACH

The system (5) can be transformed into a first order semi-explicit one, by simply defining a new set of variables by the relation $\dot{\mathbf{q}} = \mathbf{v}$,

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{M}} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \bar{\mathbf{Q}} \end{bmatrix} \tag{30}$$

$$\hat{\mathbf{M}}(\mathbf{y},\boldsymbol{\rho})\dot{\mathbf{y}} = \hat{\mathbf{Q}}(t,\mathbf{y},\boldsymbol{\rho}) \tag{31}$$

In (31) the new vector $\mathbf{y} = \left[\mathbf{q}^{\mathrm{T}}\ \mathbf{v}^{\mathrm{T}}\right]^{\mathrm{T}}$ was defined in order to lead the system from second to first order. Taking the inverse of the leading matrix in (31) the system can be expressed as a first order explicit one.

$$\dot{\mathbf{y}} = \hat{\mathbf{M}}^{-1}(\mathbf{y},\boldsymbol{\rho})\hat{\mathbf{Q}}(t,\mathbf{y},\boldsymbol{\rho}) = \mathbf{f}(t,\mathbf{y},\boldsymbol{\rho}) \tag{32}$$

The idea is to obtain again the sensitivity of the cost function (6), calculating the gradient in a diferent way. The objective function is now expressed as a function of the first order states.

$$\psi = \int_{t_0}^{t_F} g(\mathbf{y},\boldsymbol{\rho})\,\mathrm{dt} \tag{33}$$

Following the work of [7], let's consider the following Lagrangian, given by the cost function subjected to the equations of motion.

$$L(\boldsymbol{\rho}) = \int_{t_0}^{t_F} g(\mathbf{y},\boldsymbol{\rho})\,\mathrm{dt} - \int_{t_0}^{t_F} \boldsymbol{\mu}^{\mathrm{T}}(\dot{\mathbf{y}} - \mathbf{f}(t,\mathbf{y},\boldsymbol{\rho}))\,\mathrm{dt} \tag{34}$$

Where $\boldsymbol{\mu}$ is the vector of Lagrange multipliers. Applying variational calculus.

$$\begin{aligned} \delta L = &\int_{t_0}^{t_F} \left(\frac{\partial g}{\partial \mathbf{y}}\delta\mathbf{y} + \frac{\partial g}{\partial \boldsymbol{\rho}}\delta\boldsymbol{\rho}\right)\mathrm{dt} \\ &- \int_{t_0}^{t_F} \delta\boldsymbol{\mu}^{\mathrm{T}}(\dot{\mathbf{y}} - \mathbf{f}(t,\mathbf{y},\boldsymbol{\rho}))\,\mathrm{dt} \\ &- \int_{t_0}^{t_F} \boldsymbol{\mu}^{\mathrm{T}}\left(\delta\dot{\mathbf{y}} - \frac{\partial \mathbf{f}}{\partial \mathbf{y}}\delta\mathbf{y} - \frac{\partial \mathbf{f}}{\partial \boldsymbol{\rho}}\delta\boldsymbol{\rho}\right)\mathrm{dt} \end{aligned} \tag{35}$$

The parenthesis in the central term are the equations of motion, therefore if they are fulfilled in each time step, the term vanishes. For the last term, integration by parts can be applied.

$$\int_{t_0}^{t_F} \boldsymbol{\mu}^{\mathrm{T}}\delta\dot{\mathbf{y}}\mathrm{dt} = \boldsymbol{\mu}^{\mathrm{T}}\delta\mathbf{y}\Big|_{t_0}^{t_F} - \int_{t_0}^{t_F} \dot{\boldsymbol{\mu}}^{\mathrm{T}}\delta\mathbf{y}\mathrm{dt} \tag{36}$$

Therefore.

$$\begin{aligned} \delta L = &\int_{t_0}^{t_F} \left(\frac{\partial g}{\partial \mathbf{y}} + \boldsymbol{\mu}^{\mathrm{T}}\frac{\partial \mathbf{f}}{\partial \mathbf{y}} + \dot{\boldsymbol{\mu}}^{\mathrm{T}}\right)\delta\mathbf{y}\mathrm{dt} \\ &+ \int_{t_0}^{t_F} \left(\frac{\partial g}{\partial \boldsymbol{\rho}} + \boldsymbol{\mu}^{\mathrm{T}}\frac{\partial \mathbf{f}}{\partial \boldsymbol{\rho}}\right)\delta\boldsymbol{\rho}\mathrm{dt} - \\ &\boldsymbol{\mu}^{\mathrm{T}}(t_F)\,\delta\mathbf{y}(t_F) + \boldsymbol{\mu}^{\mathrm{T}}(t_0)\,\delta\mathbf{y}(t_0) \end{aligned} \tag{37}$$

In equation (37), $\delta\mathbf{y}(t_0)$ in last term is known and the previous term can be cancelled choosing $\boldsymbol{\mu}(t_F) = \mathbf{0}$. Moreover, to avoid calculating $\delta\mathbf{y}$, the first integral can be canceled by choosing $\boldsymbol{\mu}$ to be the solution of following adjoint ODE system.

$$\dot{\boldsymbol{\mu}} = -\frac{\partial \mathbf{f}}{\partial \mathbf{y}}^{\mathrm{T}}\boldsymbol{\mu} - \frac{\partial g}{\partial \mathbf{y}}^{\mathrm{T}} \tag{38}$$

$$\boldsymbol{\mu}(t_F) = 0 \tag{39}$$

The adjoint system (38), (39) is a first order linear ODE in $\boldsymbol{\mu}$, that can be integrated backwards in time from t_F to t_0 as an initial value problem.

Therefore, from equation (37) the gradient of the cost function with respect to parameters can be obtained as

$$\nabla_{\boldsymbol{\rho}}\psi = \frac{\partial \psi}{\partial \boldsymbol{\rho}}^{\mathrm{T}} = \frac{\partial \mathbf{y_0}}{\partial \boldsymbol{\rho}}^{\mathrm{T}}\boldsymbol{\mu}(t_0) + \int_{t_0}^{t_F}\left(\frac{\partial \mathbf{f}}{\partial \boldsymbol{\rho}}^{\mathrm{T}}\boldsymbol{\mu} + \frac{\partial g}{\partial \boldsymbol{\rho}}^{\mathrm{T}}\right)\mathrm{dt} \tag{40}$$

In the previous result the identity $\delta\psi = \delta L$ was used, which holds if the equations of motion are satisfied, as can be derived from (34).

In (40) and (38) the derivatives of function g are known, since the objective function has a known expression. To obtain the derivatives of $\mathbf{f}$, expression (31) can be used.

$$\hat{\mathbf{M}}\frac{\partial \mathbf{f}}{\partial \mathbf{y}}+\frac{\partial \hat{\mathbf{M}}}{\partial \mathbf{y}}\mathbf{f}=\frac{\partial \hat{\mathbf{Q}}}{\partial \mathbf{y}} \Rightarrow \frac{\partial \mathbf{f}}{\partial \mathbf{y}}=\hat{\mathbf{M}}^{-1}\left(\frac{\partial \hat{\mathbf{Q}}}{\partial \mathbf{y}}-\frac{\partial \hat{\mathbf{M}}}{\partial \mathbf{y}}\mathbf{f}\right) \quad (41)$$

$$\hat{\mathbf{M}}\frac{\partial \mathbf{f}}{\partial \boldsymbol{\rho}}+\frac{\partial \hat{\mathbf{M}}}{\partial \boldsymbol{\rho}}\mathbf{f}=\frac{\partial \hat{\mathbf{Q}}}{\partial \boldsymbol{\rho}} \Rightarrow \frac{\partial \mathbf{f}}{\partial \boldsymbol{\rho}}=\hat{\mathbf{M}}^{-1}\left(\frac{\partial \hat{\mathbf{Q}}}{\partial \boldsymbol{\rho}}-\frac{\partial \hat{\mathbf{M}}}{\partial \boldsymbol{\rho}}\mathbf{f}\right) \quad (42)$$

The derivatives $\frac{\partial \mathbf{f}}{\partial \mathbf{y}}$ and $\frac{\partial \mathbf{f}}{\partial \boldsymbol{\rho}}$ can be calculated by blocks.

$$\frac{\partial \mathbf{f}}{\partial \mathbf{y}}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \bar{\mathbf{M}}^{-1}\end{bmatrix}\left(\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\bar{\mathbf{K}} & -\bar{\mathbf{C}}\end{bmatrix}-\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \frac{\partial \bar{\mathbf{M}}}{\partial \mathbf{q}}\dot{\mathbf{v}} & \mathbf{0}\end{bmatrix}\right)= \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\bar{\mathbf{M}}^{-1}\left(\bar{\mathbf{K}}+\bar{\mathbf{M}}_{\mathbf{q}}\dot{\mathbf{v}}\right) & -\bar{\mathbf{M}}^{-1}\bar{\mathbf{C}}\end{bmatrix} \quad (43)$$

$$\frac{\partial \mathbf{f}}{\partial \boldsymbol{\rho}}=\begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{M}}^{-1}\left(\frac{\partial \bar{\mathbf{Q}}}{\partial \boldsymbol{\rho}}+\frac{\partial \bar{\mathbf{M}}}{\partial \boldsymbol{\rho}}\dot{\mathbf{v}}\right)\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{M}}^{-1}\left(\bar{\mathbf{Q}}_{\boldsymbol{\rho}}+\bar{\mathbf{M}}_{\boldsymbol{\rho}}\dot{\mathbf{v}}\right)\end{bmatrix} \quad (44)$$

In (43), $\bar{\mathbf{K}}=-\frac{\partial \bar{\mathbf{Q}}}{\partial \mathbf{q}}$ is given by equation (16), $\bar{\mathbf{C}}=-\frac{\partial \bar{\mathbf{Q}}}{\partial \mathbf{v}}=-\frac{\partial \bar{\mathbf{Q}}}{\partial \dot{\mathbf{q}}}$ is given by equation (17), $\bar{\mathbf{Q}}_{\boldsymbol{\rho}}$ is given by equation (18) and the terms $\bar{\mathbf{M}}_{\mathbf{q}}\dot{\mathbf{v}}=\bar{\mathbf{M}}_{\mathbf{q}}\ddot{\mathbf{q}}$ and $\bar{\mathbf{M}}_{\boldsymbol{\rho}}\dot{\mathbf{v}}=\bar{\mathbf{M}}_{\boldsymbol{\rho}}\ddot{\mathbf{q}}$ are given by equations (19) and (20) respectively.

Finally, it turns out that the same derivatives employed the direct sensitivity approach (equations (13) and (10)) are necessary for the adjoint sensitivity approach (equations (38) and (40)).

VALIDATION OF THE COMPUTED SENSITIVITIES

Several approaches were used to make sure that the formulations proposed compute the sensitivities correctly and that all the derivatives proposed are correct. It is important to remark, that any mistake, even small, in the derivatives involved in the direct or adjoint approaches can lead to completely different results in the sensitivities computed.

The validation proposed and implemented here included several strategies:

1. Compare the results of direct and adjoint sensitivity approaches. They should be equal within the truncation error.
2. Compute the sensitivities using a third party code: FATODE [8].
3. Use *real* finite differences to approximate whole sensitivities or individual derivatives. This approach can be very inaccurate or even completely useless, as proved in the numerical examples.
4. Use *complex* finite differences to approximate whole sensitivities or individual derivatives. This approach is much more reliable than the previous one, but more complex to implement.

Compute the sensitivities using FATODE

The code computes the direct dynamics and sensitivities using adjoint techniques. Since the derivatives are provided by the user, the comparison can only detect errors in the algorithms but not in the derivatives.

The forward, adjoint, and tangent linear integration of ODEs (FATODE) is a library which provides explicit/implicit Runge-Kutta and Rosenbrock integrators for nonstiff and stiff ODEs. The forward model can solve ODE systems. The tangent linear model and the discrete adjoint model are used by the integrators in FATODE to perform sensitivity analysis. To use the integrators in FATODE for the forward simulations, two basic functions, $\mathbf{f}(t,\mathbf{y},\boldsymbol{\rho})$ and $\frac{\partial \mathbf{f}}{\partial \mathbf{y}}$, are required. Besides, the objective function ψ, which is defined in (45), and several additional functions are also required for sensitivity analysis. In addition, for sensitivity analysis, an additional function is required by the integrators to initialize the adjoint variable $\boldsymbol{\lambda}_s$ and $\boldsymbol{\mu}_s$ before the backward simulation. The functions to define and their connections with the equations of this work, are the following.

ψ: the objective function, which is defined as follows:

$$\psi=r\left(\mathbf{y}\left(t_F\right),\boldsymbol{\rho}\right)+\int_{t_0}^{t_F} g\left(\mathbf{y},\boldsymbol{\rho}\right)\mathrm{d}t \quad (45)$$

$\mathbf{f}(t,\mathbf{y},\boldsymbol{\rho})$: the right-hand side function of the ODE, which is defined in (32).

$\frac{\partial \mathbf{f}}{\partial \mathbf{y}}$: the Jacobian of the right-hand side function with respect to the state vector, which is defined in (43).

$\frac{\partial \mathbf{f}}{\partial \boldsymbol{\rho}}$: the Jacobian of the right-hand side function with respect to the parameters, which is defined in (44).

$g(\mathbf{y},\boldsymbol{\rho})$: the function which is defined in (45).

$\frac{\partial g}{\partial \boldsymbol{\rho}}$: the partial derivative of $g(\mathbf{y},\boldsymbol{\rho})$ with respect to the parameters $\boldsymbol{\rho}$.

$\frac{\partial g}{\partial \mathbf{y}}$: the partial derivative of $g(\mathbf{y},\boldsymbol{\rho})$ with respect to the state vector $\mathbf{y}$.

$\boldsymbol{\lambda}_s$: the sensitivities of the objective function ψ with respect to the initial conditions, which is $\boldsymbol{\mu}$ from (38). $\boldsymbol{\lambda}_s$ should be initialized to be $\mathbf{0}$ from (43).

$\boldsymbol{\mu}_s$: the sensitivities of the objective function ψ with respect to the parameters. In this paper, $\boldsymbol{\mu}_s$ is the output of the sensitivities. It should be initialized to be $\mathbf{0}$

These functions are provided to the adjoint fully implicit Runge-Kutta solver to compute the forward solution and the sensitivities.

Real and complex differences approximation

Although impractical from the computational point of view, the finite differences approximation can be very useful to detect errors in the derivatives. The first order approximation for the derivatives with real perturbations, read as follows.

$$\frac{\mathrm{d}\psi}{\mathrm{d}\boldsymbol{\rho}_k} = \frac{\psi(\boldsymbol{\rho}+\delta\mathbf{e}_k)-\psi(\boldsymbol{\rho})}{\delta} \tag{46}$$

The truncation error in this case is $\vartheta(h)$, where h is the time-step, so it can be controlled decreasing it. Nevertheless, small h results in loss-of-significance (cancellation) errors due to the substraction. This fact can make this derivatives completely useless or untrustworthy, as will be shown in the next section.

The first order approximation for the derivatives with complex perturbations is the following.

$$\frac{\mathrm{d}\psi}{\mathrm{d}\boldsymbol{\rho}_k} = \frac{\Im\left(\psi(\boldsymbol{\rho}+i\delta\mathbf{e}_k)\right)}{\delta} \tag{47}$$

Where i is the imaginary unit and $\Im$ is the imaginary part of a complex number. The approach is much more trustworthy than the previous one, since there are no loss-of-significance errors involved in the calculation of the approximation, because there are not substractions in the imaginary parts and therefore the increments can be chosen arbitrarily small. The practical difficulty to apply complex finite differences is that not all codes can be changed easily to accommodate complex arithmetic. Special attention should be paid to the third party functions involved in the code (*transpose* functions, *norm* functions, numerical integrator chosen, etc).

This approach was used in this study to validate all the derivatives and results presented.

NUMERICAL EXPERIMENTS

Five bar mechanism

The mechanism chosen to test the formulations proposed in the paper is the five bar mechanism with 2 degrees of freedom shown in Fig.1. The five bars are constrained by five revolute joints located in points A, 1, 2, 3 and B. The five bars are constrained by five revolute joints located in points A, 1, 2, 3 and B.

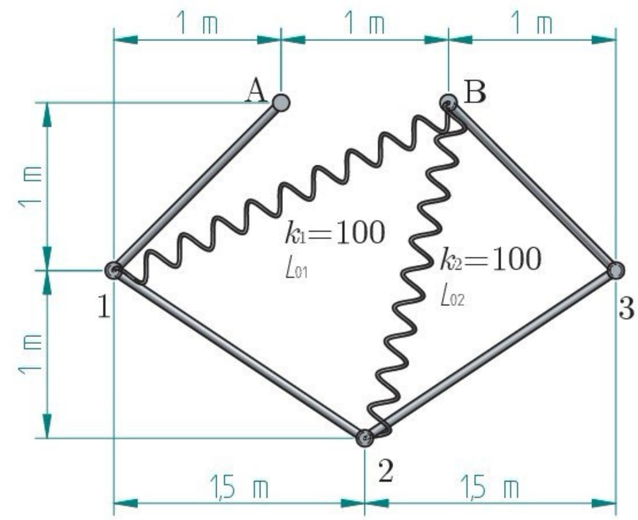

FIGURE 1. FIVE BAR MECHANISM

The masses of the bars are $m_1 = 1\ kg$, $m_2 = 1.5\ kg$, $m_3 = 1.5\ kg$, $m_4 = 1\ kg$ and the polar moments of inertia are calculated under the assumption of a uniform distribution of mass. The mechanism is subjected to the action of gravity and two elastic forces coming from the springs. The stiffness coefficients of the springs are $k_1 = k_2 = 100\,N/m$ and their natural lengths are initially chosen $L_{01} = \sqrt{2^2+1^2}\ m$ and $L_{02} = \sqrt{2^2+0.5^2}\ m$, coincident with the initial configuration shown in Fig.1.

The mechanism can be balanced by properly selecting the two parameters $\boldsymbol{\rho}^{\mathrm{T}} = [L_{01}, L_{02}]$. Of course the problem can be solved by means of the static equations but the aim here is doing so by dynamical optimization.

The objective is to keep the mechanism still in the initial position which can be represented mathematically by the following objective function.

$$\psi = \int_{t_0}^{t_F} (\mathbf{r}_2-\mathbf{r}_{20})^{\mathrm{T}}(\mathbf{r}_2-\mathbf{r}_{20})\,\mathrm{d}t \tag{48}$$

Where $\mathbf{r}_2$ is the global position of the point 2 and $\mathbf{r}_{20}$ is the initial position of the same point.

The condition to obtain the minimum is the following.

$$\nabla_{\boldsymbol{\rho}}\psi = \mathbf{0} \tag{49}$$

The gradient (49) was obtained by the following approaches:

1. Direct sensitivity: using equation (10).
2. Adjoint sensitivity: using equation (40).
3. Adjoint sensitivity with FATODE.

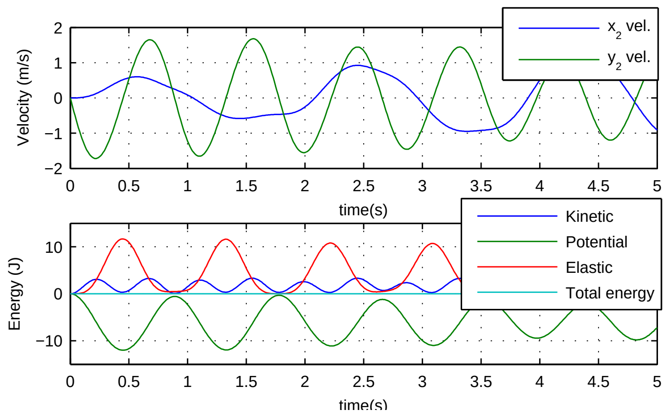

FIGURE 2. MECHANISM RESPONSE.

4. Numerical sensitivity with real perturbations.
5. Numerical sensitivity with complex perturbations.

The response of the system is shown in Fig.2 for a 5 seconds simulation. The upper plot represents the horizontal and vertical velocities of the point 2 while the lower one represents the energy taking as reference for the potential energy the initial configuration of the system.

The results for the sensitivities with the mentioned methods are presented in Table 1.

TABLE 1. RESULTS FOR THE FIVE BAR MECHANISM.

Approach	Parameters	$d\psi/dL_{01}$	$d\psi/dL_{02}$
1: Direct	$h = 10^{-2}s$	-4.2305	3.2154
2: Adjoint-1	$h = 10^{-2}s$	-4.2299	3.2134
4: FATODE	$Tol = 10^{-3}$	-4.2257	3.2077
5: Num. diff. real	$\delta = 10^{-7}m$	-9.7390	-4.0344
6: Num. diff. complex	$\delta/i = 10^{-7}m$	-4.2288	3.2116

As can be seen in Table 1, all the approaches, except the numerical sensitivities with real perturbations, offer similar results which guarantees that the schemes proposed are correct. The numerical sensitivities with real perturbations are not reliable if accurate results for the sensitivities are important for the application to tackle. Given the simplicity of the system proposed, definitive conclusions in terms of efficiency cannot be stated.

The computed sensitivities can be employed for the optimization proposed. All the methods perform similar to solve

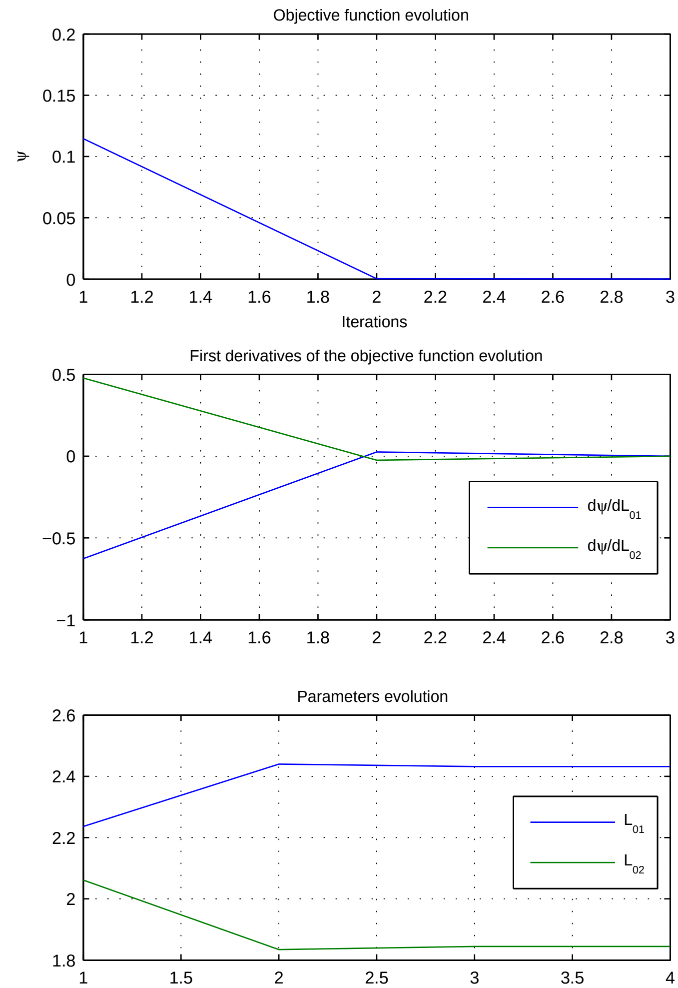

FIGURE 3. OBJECTIVE FUNCTION, GRADIENT AND PARAMETERS EVOLUTION.

the optimization problem. In this case the simulation time was reduced to $1s$ and the results for the objective function, derivatives and parameters are presented in Fig.3 for the *Adjoint-1* approach. The plots for the direct approach coincide with the ones presented and they are not presented for clarity.

The optimization converges in three iterations, but in one is almost done. It is important to remark that approximate derivatives can be used to calculate the gradient and the optimization would converge at a lower pace.

Another important remark is that the tolerances in the solution of the forward dynamics are very important in order to obtain stable solutions for the TLM and adjoint ODEs, both of them strongly depends on the solution of the dynamics.

Suspension system

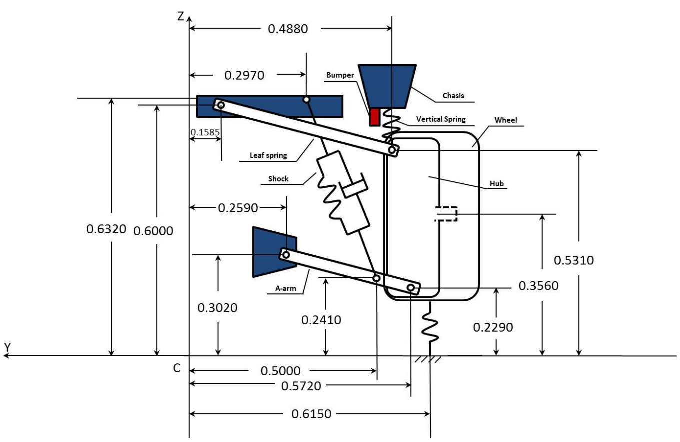

FIGURE 4. REAR DOUBLE WISHBONE SUSPENSION

The methods described above are used to calculate the sensitivity of the Iltis vehicle rear suspension [9]. The Iltis vehicle was proposed as a benchmark problem by the European automobile industryto check multibody dynamic codes. Its rear suspension can be modeled as a double wishbone suspension, which is an independent suspension which connects the wheel hub by two arms (Leaf spring and Lower A-Arm). As shown in Figure 4, the chassis, A-Arms and the wheel hub are connected through the kinematic structure by revolute joints. The chassis is modeled as a rigid body with two degrees of freedom, x (longitudinal) and z (vertical) translation. Additionally to those two degrees of freedom, the suspension has one additional degree of freedom and the wheel rotation is not considered here. The contact between the tire and the ground is modeled by a spring-damper acting in the vertical direction. Since only one wheel is considered, the motion of the chassis was properly constrained to allow only the degrees of freedom indicated before, moreover, in order to approximate the real behavior, only one quarter of the total chassis mass is considered.

The center of mass of the vehicle cabin is at CGc in Fig. 5. It defines the nominal configuration of the vehicle which is not in equilibrium.

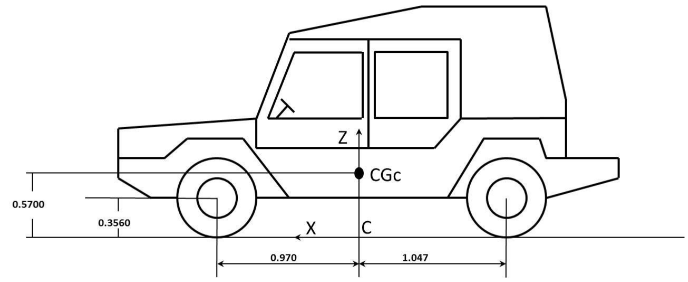

FIGURE 5. The Bombardier Iltis vehicle

The coordinates of the key points are given in Table 2.

TABLE 2. Positions of nodes of right rear suspension(origin C, Fig. 5)

Point description	x [m]	y [m]	z [m]
wheel center	-1.047	-0.615	0.356
A-arm connection to hub carrier	-1.047	-0.572	0.229
A-arm connection to cabin	-1.047	-0.259	0.302
leaf spring connection to hub carrier	-1.047	-0.488	0.531
leaf spring connection to cabin	-1.047	-0.1585	0.600
damper connection to A-arm	-0.972	-0.500	0.241
damper connection to cabin	-0.972	-0.297	0.632

The mass, center of mass and inertia values of bodies are given in Table 3 and Table 4.

TABLE 3. Positions of centers of mass (origin C. Fig. 5)

Body	$Coord.ofCG(m)$		
	x	y	z
cabin	0	0	0.57
right rear wheel with hub and brake assembly	-1.047	-0.615	0.356
right rear A-arm	-1.047	-0.4155	0.2655

TABLE 4. Mass and moments of inertia of bodies

Body	Mass(kg)	$I_{xx}(kgm^2)$	$I_{yy}(kgm^2)$	$I_{zz}(kgm^2)$
cabin	1260	130	1620	1670
right rear wheel with hub and brake assembly	57.35	1.2402	1.908	1.2402
right rear A-arm	6.0	0.052099	0.023235	0.068864

The system is released with an initial velocity of 5 m/s in the longitudinal direction, while the velocities of the two remaining degrees of freedom are set to zero. Since the initial vertical position doesn't correspond to the static equilibrium in the vertical direction, the suspension oscillates until the static equilibrium is reached. At a distance of 6 m ahead from the initial position in the x (longitudinal) direction, a step of 10 cm is placed. After 5 s. the suspension drops down the step and oscillates until the static equilibrium in the vertical direction is reached again.The response of the suspension system traveling along the step is shown in Fig. 6: the upper plot represents the vertical coordinate of the CDG of the chassis while the lower one shows the vertical coordinate of the CDG of the wheel.

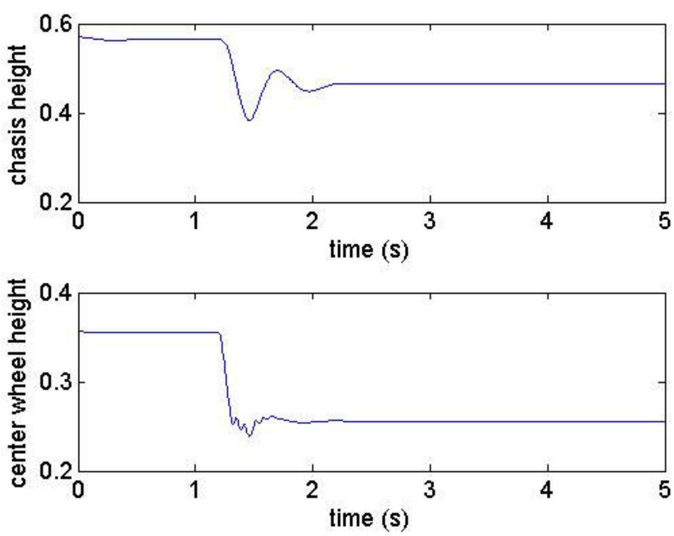

FIGURE 6. RESPONSE OF THE SUSPENSION MODEL

The objective function is the integral of the vertical velocity square of the chassis along the simulation and the design parameter chosen is the stiffness of the spring attached to the leaf spring $\boldsymbol{\rho} = k$.

$$\psi = \int_0^t (\dot{z}_{chassis})^2 \, dt \tag{50}$$

Table 5 summarizes the sensitivities obtained for the suspension system with different final simulation times. The results obtained correspond to the adjoint sensitivity method and the numerical validation against complex perturbation sensitivities is also provided in order to check the correctness of the results provided.The results are very close and the differences are due to the fact that different time-stepping schemes were employed for the adjoint and the complex perturbations.

TABLE 5. RESULTS FOR THE SUSPENSION SYSTEM.

Time	Adjoint	Complex perturbation
1 s.	-1.48710^{-07}	-1.48710^{-07}
1.5 s.	1.68510^{-06}	1.70510^{-06}
3 s.	4.73010^{-06}	4.74210^{-06}
5 s.	4.73010^{-06}	4.74210^{-06}

This results could be used in a latter stage to optimize the behavior of the suspension system under the parameters selected.

CONCLUSIONS

In this paper, two approaches for the sensitivity analysis of multibody systems based on penalty formulations were developed: the direct sensitivity approach and the adjoint sensitivity approach.

All the results were tested and validated comparing the two approaches between them, comparing with a third party software and comparing with numerical results obtained by means of real and complex perturbations. The complex perturbation approach showed to be much more reliable than its real counterpart.

The results obtained were used for the sensitivity analysis of two multibody systems, one of them academic and the other one industrial: a five bar mechanism and a vehicle suspension system.

ACKNOWLEDGMENT

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