Direct differentiation of state-space equations of motion in natural coordinates

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Abstract

This article deals with the theoretical and practical comparison of different state-of-the-art techniques for the computation of sensitivities in multibody systems. Specifically, we study state-space multibody formulations in natural coordinates applied to medium-large mechanical systems such as road vehicles. The main family of differentiation techniques under study is the direct differentiation method (DDM) [1, 2], both from *manual* [3] and *automatic* [4] differentiation perspectives.

Let f be the number of degrees of freedom (DOFs) of the system, n the number of dependent coordinates and $\mathbf{b} \in \mathbb{R}^p$ the vector of design parameters. The state-space motion differential equations in natural or fully Cartesian coordinates, according to the matrix-R method [5], can be written as:

$$\hat{\mathbf{M}}(\mathbf{z},\mathbf{b})\,\ddot{\mathbf{z}}(t) = \hat{\mathbf{Q}}(t,\mathbf{z},\dot{\mathbf{z}},\mathbf{b}) \tag{1a}$$

$$\hat{\mathbf{M}} \equiv \mathbf{R}^T\mathbf{M}\mathbf{R} \tag{1b}$$

$$\hat{\mathbf{Q}} \equiv \mathbf{R}^T(\mathbf{Q}-\mathbf{M}\mathbf{S}\mathbf{c}) \tag{1c}$$

where $\mathbf{q} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{R}^f$ are, respectively, the vectors of dependent and independent coordinates; $\mathbf{R} \in \mathbb{R}^{n\times f}$ and $\mathbf{S} \in \mathbb{R}^{n\times m}$ can be calculated as explained in [5]; $\mathbf{M}(\mathbf{q},\mathbf{b}) \in \mathbb{R}^{n\times n}$ and $\mathbf{Q}(t,\mathbf{q},\dot{\mathbf{q}},\mathbf{b}) \in \mathbb{R}^n$ are, respectively, the mass matrix and vector of generalized forces in dependent coordinates; $\mathbf{c} \equiv \mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = -\dot{\mathbf{\Phi}}_{\mathbf{q}}\dot{\mathbf{q}} - \dot{\mathbf{\Phi}}_t$; and $\mathbf{\Phi}(t,\mathbf{q},\mathbf{b}) \in \mathbb{R}^m$ is the vector of constraint equations.

Sensitivity equations can then be obtained through the DDM [1, 2]. Differentiating the equations of motion (1) with respect to the design parameters and rearranging we obtain:

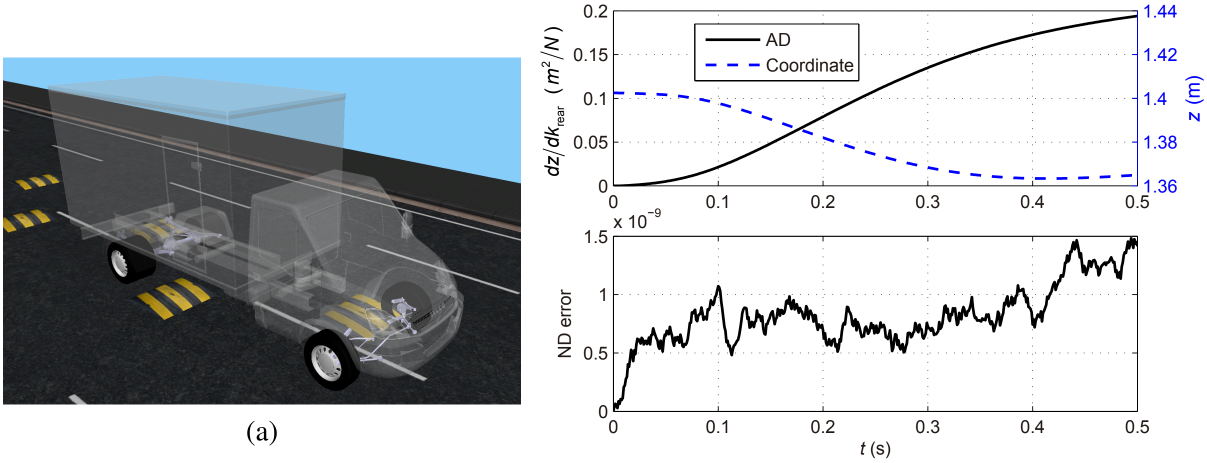
$$\hat{\mathbf{M}}\ddot{\mathbf{z}}_{\mathbf{b}} + \hat{\mathbf{C}}\dot{\mathbf{z}}_{\mathbf{b}} + \left(\hat{\mathbf{K}} + \hat{\mathbf{M}}_{\mathbf{z}}\ddot{\mathbf{z}}\right)\mathbf{z}_{\mathbf{b}} = \hat{\mathbf{Q}}_{\mathbf{b}} - \hat{\mathbf{M}}_{\mathbf{b}}\ddot{\mathbf{z}} \tag{2a}$$

$$\mathbf{z}_{\mathbf{b}}(t_0) = \mathbf{z}_{\mathbf{b}0} \tag{2b}$$

$$\dot{\mathbf{z}}_{\mathbf{b}}(t_0) = \dot{\mathbf{z}}_{\mathbf{b}0} \tag{2c}$$

where $\hat{\mathbf{K}} \equiv -\partial\hat{\mathbf{Q}}/\partial\mathbf{z}$, $\hat{\mathbf{C}} \equiv -\partial\hat{\mathbf{Q}}/\partial\dot{\mathbf{z}}$ and $\hat{\mathbf{Q}}_{\mathbf{b}} \equiv -\partial\hat{\mathbf{Q}}/\partial\mathbf{b}$ are the derivatives of the projected forces, $\hat{\mathbf{Q}}$; and tensor-vector product $\hat{\mathbf{M}}_{\mathbf{z}}\ddot{\mathbf{z}} \equiv \hat{\mathbf{M}}_{\mathbf{z}} \otimes \ddot{\mathbf{z}}$ represents the derivatives of vector $\hat{\mathbf{M}}\ddot{\mathbf{z}}$ when $\ddot{\mathbf{z}}$ is considered constant. This technique results in one set of ordinary differential equations (ODEs) per design parameter. These sets of ODEs can be integrated in a standard way, together with the set of ODEs in Equation (1), to find independent *state sensitivities*, $\mathbf{z}_{\mathbf{b}}$. Then, dependent sensitivity matrices $\mathbf{q}_{\mathbf{b}}$, $\dot{\mathbf{q}}_{\mathbf{b}}$ and $\ddot{\mathbf{q}}_{\mathbf{b}}$ allow one to compute the vector of *design sensitivities* (or gradient) $\Psi_{\mathbf{b}}$, where $\Psi = \Psi(t,\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\mathbf{b})$ is a user-defined analytical objective function.

A 17-DOF box truck model with realistic geometry and suspension compoments (see Figure 1(a)) is used as a numerical example. Sensitivity equations (2) are assembled using manual and automatic differentiation techniques, and then numerically integrated using standard integrators. An example state sensitivity is shown in Figure 1(b). The results are thoroughly compared in terms of accuracy and efficiency.



(a)

(b)

Figure 1: (a) 17-DOF box truck model used to validate the presented methods; (b) Sensitivity of the vertical position (z) w.r.t. the rear axle stiffness (k_{rear}) — AD stands for automatic differentiation and ND for numerical differentiation.

This work constitutes a step forward in a series of recent articles about the general-purpose sensitivity analysis of multibody systems [3, 4, 6]. The main contribution of this work is the comparison and validation, through a medium-large numerical example, of direct differentiation techniques for the computation of sensitivities in dynamic mechanical systems. Such a detailed survey, based on in-house multibody software, has never been presented in the literature.

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