Optimal Control of Multibody Systems in Reduced Space ODE Formulation

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Abstract: *Adjoint-based optimization of design and control for multibody systems has been the subject of rigorous research in recent years[1]. The novel contribution of this article is in the development of an efficient adjoint methodology for optimal control of rigid multibody systems by exploiting the technique of computing control in a reduced temporal space using appropriate basis functions. Use of basis functions yields a substantial reduction in the required number of control parameters which drastically improves computational performance. This methodology also implements a robust ODE formulation which was recently developed for multibody systems to overcome the drawbacks of DAE formulations. The closed-form equations for Jacobians of the state transition and objective functions which are required for adjoint computation have been derived for the ODE formulation using tensor algebra. Since the algorithm does not distinguish between control and design parameters, it enables an integrated optimization approach, wherein the design and control can be simultaneously optimized.*

The problem statement for optimal control of a generic dynamic system can be stated in the form of an constrained optimization case study [2].

$$\min_{u} \quad \boldsymbol{\psi} = \mathbf{w}(\mathbf{x}, \boldsymbol{\rho})|_{t_f} + \int_{t_0}^{t_f} \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}, \boldsymbol{\rho}, \mathbf{u})\, dt \qquad \text{s.t.} \quad \dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \boldsymbol{\rho}, \mathbf{u}\right) \tag{1}$$

Taking infinitesimal variations of the Lagrangian we have

$$\delta\mathcal{L} = \mathbf{w}_{\boldsymbol{\rho}}\delta\boldsymbol{\rho}\big|_{t_f} + \boldsymbol{\mu}^{\mathrm{T}}\delta\mathbf{x}|_{t_0} + \int_{t_0}^{t_f} \Big[\left(\mathbf{g}_{\boldsymbol{\rho}} + \mathbf{g}_{\dot{\mathbf{x}}}\mathbf{f}_{\boldsymbol{\rho}} + \boldsymbol{\mu}^{\mathrm{T}}\mathbf{f}_{\boldsymbol{\rho}}\right)\delta\boldsymbol{\rho} + \left(\mathbf{g}_{\mathbf{u}} + \mathbf{g}_{\dot{\mathbf{x}}}\mathbf{f}_{\mathbf{u}} + \boldsymbol{\mu}^{\mathrm{T}}\mathbf{f}_{\mathbf{u}}\right)\delta\mathbf{u}\Big]\, dt \tag{2}$$

yielding the adjoint equation along its boundary condition at the final time as $\dot{\boldsymbol{\mu}} = -\mathbf{f}_{\mathbf{x}}^{\mathrm{T}}\left(\boldsymbol{\mu} + \mathbf{g}_{\dot{\mathbf{x}}}^{\mathrm{T}}\right) - \mathbf{g}_{\mathbf{x}}^{\mathrm{T}}$ and $\boldsymbol{\mu}|_{t_f} = \mathbf{w}_{\mathbf{x}}^{\mathrm{T}}|_{t_f}$. Now, we have $\delta\mathcal{L} = \langle\nabla_{\boldsymbol{\rho}}\mathcal{L},\, \delta\boldsymbol{\rho}\rangle + \langle\nabla_{\mathbf{u}}\mathcal{L},\, \delta\mathbf{u}\rangle$ where

$$\nabla_{\boldsymbol{\rho}}\mathcal{L}^{\mathrm{T}} = \mathbf{w}_{\boldsymbol{\rho}}\big|_{t_f} + \boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}_0' + \int_{t_0}^{t_f} \left(\mathbf{g}_{\boldsymbol{\rho}} + \mathbf{g}_{\dot{\mathbf{x}}}\mathbf{f}_{\boldsymbol{\rho}} + \boldsymbol{\mu}^{\mathrm{T}}\mathbf{f}_{\boldsymbol{\rho}}\right) dt \qquad \nabla_{\mathbf{u}}\mathcal{L}^{\mathrm{T}} = \mathbf{g}_{\mathbf{u}} + \left(\boldsymbol{\mu}^{\mathrm{T}} + \mathbf{g}_{\dot{\mathbf{x}}}\right)\mathbf{f}_{\mathbf{u}} \tag{3}$$

The control is parameterized as an explicit function of time using basis functions represented by $\mathbf{B}_i(t)$ and their corresponding time-invariant coefficients $\mathbf{c}_i$ where $i = 1, 2, ..., r$. Thus we have $\mathbf{u}(t) = \sum_{i=1}^{r} \mathbf{c}_i\, \mathbf{B}_i(t) \implies \delta\mathbf{u}(t) = \sum_{i=1}^{r} \delta\mathbf{c}_i\, \mathbf{B}_i(t)$. Now

$$\delta_{\mathbf{u}}\mathcal{L} = \sum_{i=1}^{r} \int_{t_0}^{t_f} \left(\nabla_{\mathbf{u}}\mathcal{L} \cdot \mathbf{B}_i\right) \cdot \delta\mathbf{c}_i\, dt \qquad \implies \nabla_{\mathbf{c}_i}\mathcal{L}^{\mathrm{T}} = \int_{t_0}^{t_f} \nabla_{\mathbf{u}}\mathcal{L}^{\mathrm{T}} \cdot \mathbf{B}_i\, dt \tag{4}$$

The ODE formulation [3] for multibody systems in independent coordinates $\mathbf{v}$ in the centroidal form is given by $\left(\mathbf{D}^{\mathrm{T}}\mathbf{M}\mathbf{D}\right)\ddot{\mathbf{v}} = \mathbf{D}^{\mathrm{T}}\left(\mathbf{M}\mathbf{U}\mathbf{B}\boldsymbol{\gamma} + \mathbf{Q}\right)$ where $\mathbf{B} = \left(\boldsymbol{\Phi}_{\mathbf{q}}\mathbf{U}\right)^{-1}$, $\mathbf{D} =$

$\left[\mathbf{I}-\mathbf{UB\Phi_q}\right]\mathbf{V}$, $\mathbf{U}=\mathbf{\Phi_q}|_{\mathbf{q}_0}$ and $\mathbf{V}$ is the null space of $\mathbf{U}^{\mathrm{T}}$. The identities relating the generalized coordinates $\mathbf{q}$ and $\mathbf{v}$ are $\mathbf{q_v}=\dot{\mathbf{q}}_{\dot{\mathbf{v}}}=\mathbf{D}$, $\dot{\mathbf{q}}_{\mathbf{v}}=\mathbf{D_q}\dot{\mathbf{v}}\mathbf{D}$, $\mathbf{D_q}=-\left(\mathbf{UB_q\Phi_q}+\mathbf{UB\Phi_{qq}}\right)\mathbf{V}$, $\mathbf{B_q}=-\mathbf{B\Phi_{qq}UB}$. Thus the Jacobians of state transition matrix can be computed as

$$\mathbf{f}_{\dot{\mathbf{v}}}=\begin{bmatrix}\mathbf{D}^{\mathrm{T}}\mathbf{MD} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}^{-1}\begin{Bmatrix}\mathbf{D}^{\mathrm{T}}\left(\mathbf{MUB}\boldsymbol{\gamma}_{\dot{\mathbf{v}}}+\mathbf{Q}_{\dot{\mathbf{v}}}\right)\\ \mathbf{I}\end{Bmatrix}\quad \mathbf{f_u}=\begin{bmatrix}\mathbf{D}^{\mathrm{T}}\mathbf{MD} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}^{-1}\begin{Bmatrix}\mathbf{D}^{\mathrm{T}}\mathbf{Q_u}\\ \mathbf{0}\end{Bmatrix}$$

$$\mathbf{f_v}=-\begin{bmatrix}\mathbf{D}^{\mathrm{T}}\mathbf{MD} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}^{-1}\begin{bmatrix}\left(\mathbf{D}^{\mathrm{T}}\mathbf{MD}\right)_{\mathbf{v}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{f}+\begin{bmatrix}\mathbf{D}^{\mathrm{T}}\mathbf{MD} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}^{-1}\begin{Bmatrix}\left[\mathbf{D}^{\mathrm{T}}\left(\mathbf{MUB}\boldsymbol{\gamma}+\mathbf{Q}\right)\right]_{\mathbf{v}}\\ \mathbf{0}\end{Bmatrix}$$

A case study was implemented using the proposed methodology on a 1 DOF undamped spatial pendulum starting in an unstable configuration ($x=0.1$ m) and moving under the action of gravity. External forces act at C.G. of the pendulum and the objective function is chosen such that the optimization algorithm tries to bring the pendulum to rest. This can be done by using penalties $\mathbf{F}$, $\mathbf{S}$, and $\mathbf{R}$ which need to be symmetric positive semi-definite real matrices and a state error function $\mathbf{e}$ that needs to be minimized.

$$\psi=\mathbf{e}^{\mathrm{T}}\mathbf{Fe}|_{t_f}+\int_{t_0}^{t_f}\left(\mathbf{e}^{\mathrm{T}}\mathbf{Se}+\mathbf{u}^{\mathrm{T}}\mathbf{Ru}\right)\,dt=10^8x^2|_{t_f}+\int_{t_0}^{t_f}\left(2\times10^8x^2+u^2\right)\,dt \qquad (6)$$

15 linear basis functions were used for this study. The results have been shown in Figure 1.

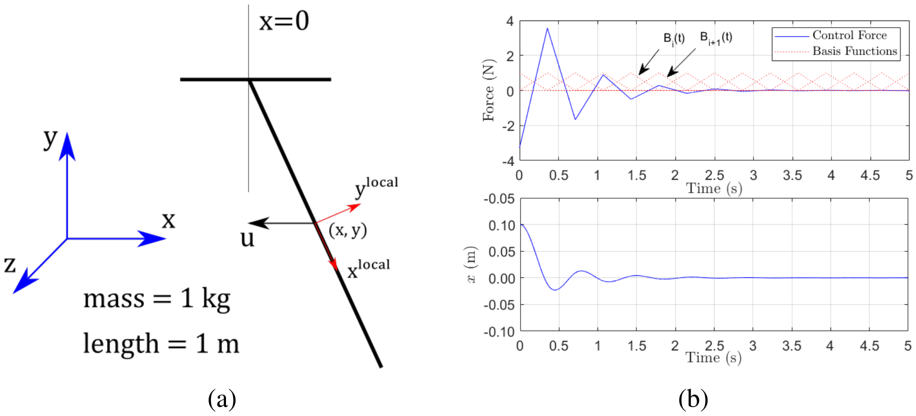

Figure 1: (a) Schematic. (b) Time History.

Conclusions

An efficient ODE-based adjoint methodology for optimization of multibody systems was presented and implemented on 1 DOF multibody system. For full-space control, typically the number of control parameters is the number of timesteps in the simulation. The reduced space technique substantially reduces the number of required parameters by using basis functions.

References

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